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A simplified formulation of adhesion problems with elastic plates

By Carmel Majidi[1] and George G. Adams[2,*]

[1]*Princeton Institute for the Science and Technology of Materials (PRISM), Princeton University, Princeton, NJ 08544, USA*
[2]*Department of Mechanical and Industrial Engineering, Northeastern University, Boston, MA 02115, USA*

The solution of adhesion problems with elastic plates generally involves solving a boundary-value problem with an assumed contact area. The contact region is then found by minimizing the total potential energy with respect to the contact area (i.e. the contact radius for the axisymmetric case). Such a procedure can be extremely long and tedious. Here, we show that the inclusion of adhesion is equivalent to specifying a discontinuous internal bending moment at the contact region boundary. The magnitude of this moment discontinuity is related to the work of adhesion and flexural rigidity of the plate. Such a formulation can greatly reduce the algebraic complexity of solving these problems. It is noted that the related plate contact problems without adhesion can also be solved by minimizing the total potential energy. However, it has long been recognized that it is mathematically more efficient to find the contact area by specifying a continuous internal bending moment at the boundary of the contact region. Thus, our *moment discontinuity method* can be considered to be a generalization of that procedure which is applicable for problems with adhesion.

Keywords: adhesion; contact mechanics; plate; thin-film

1. Introduction

Problems involving adhesion of elastic plates are important in a variety of applications. An important process in the semiconductor industry is wafer bonding. Wafers are typically bonded together by applying a load at the centre which causes the bond front to spontaneously propagate radially outward to the wafer edge (Turner & Spearing 2002), i.e. to zip shut. This process is driven by the combination of applied pressure and the surface interactions due to weak interatomic forces, such as van der Waals forces. A permanent bond is created by the formation of covalent bonds which normally occurs at a high temperature. Turner & Spearing (2002) modelled this process using elastic plate theory and minimizing the total potential energy with respect to the bond radius. An experimental investigation (Turner & Spearing 2005) verified the results of their earlier paper.

* Author for correspondence (adams_g@coe.neu.edu).

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Nanomanufacturing is a relatively new field which is of great current interest. One potential method of nanomanufacturing uses self-assembly of nanoelements on a template followed by the transfer of these elements from the template to the device wafer (Pamp & Adams 2007). However, wafers typically have waviness or bow that can lead to surface deviations of many micrometres. Thus, an outside force, or pressure, is required to bring these surfaces into intimate contact to accomplish the transfer of the nanoelements. This process is similar to that used in wafer bonding. However, unlike with semiconductors, it is very important to avoid bonding of the wafers. Thus, the elastic energy stored during the deformation must be sufficient to separate the surfaces after the pressure is removed. Both cylindrical bending and axisymmetric deformation were considered by Pamp & Adams (2007). The axisymmetric problem was solved by minimizing the total potential energy. That procedure was extremely lengthy and required the use of a symbolic interpreter language to perform the algebra. In the analysis of cylindrical bending, it was shown that for the particular cases studied, the effect of adhesion is equivalent to a discontinuity in the internal bending moment at the contact boundary.

There is also considerable interest in bio-inspired adhesives, in particular in the peeling or attachment of plate-like spatulae or lamellae of wall-climbing insects and lizards. These latter structures are treated as a thin elastic plate in contact with a rigid, non-flat substrate (Persson & Gorb 2003; Majidi & Fearing 2008). In biology, cell adhesion is also an active area of study and has been addressed by various plate theories and adhesion models (Seifert 1991; Rosso *et al.* 2000; Wan & Liu 2001).

In general, the method used to solve adhesion problems with elastic plates involves solving a boundary-value problem with an assumed contact area. The actual contact region is then found by minimizing the total potential energy (which includes the work of adhesion) with respect to the contact area (i.e. the contact radius for an axisymmetric problem). This procedure can be extremely cumbersome and tedious. In this paper, we show that the inclusion of adhesion is equivalent to specifying a discontinuous internal bending moment at the contact region boundary. The magnitude of this moment discontinuity is shown to be equal to the square root of twice the product of the work of adhesion and the plate flexural rigidity. Such a formulation can lead to an enormous reduction in the algebraic complexity of solving adhesion problems with elastic plates.

It is noted that related contact problems without adhesion can also be solved by minimizing the total potential energy (which in this case does not include the work of adhesion). However it has long been recognized that it is mathematically more efficient to find the contact area by specifying a continuous internal bending moment at the boundary of the contact region (Timoshenko & Woinowsky-Kreiger 1959, pp. 308–313; Benson 1991). Thus, our *moment discontinuity method* (MDM) constitutes a generalization of that procedure which is applicable for contact problems with adhesion. The use of the method is illustrated with some example problems.

2. Model

Suppose that an elastic, thin-walled, axisymmetric body (e.g. hollow sphere, a bowl or a circular disk) makes point contact with a rigid, axisymmetric surface and that the axis of symmetry for the two bodies coincide. As illustrated in

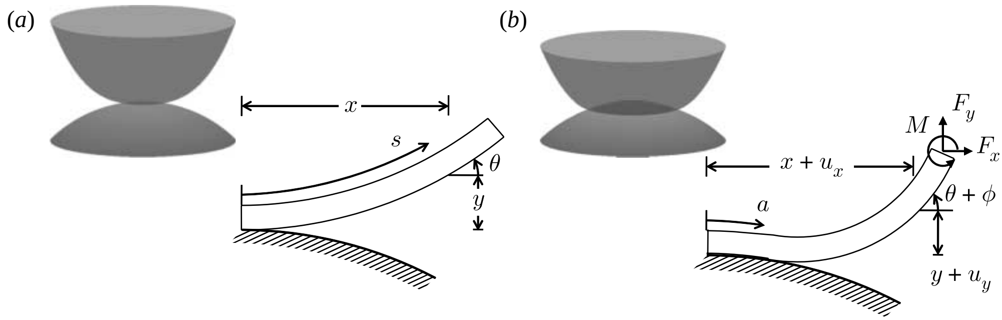

Figure 1. An initially curved plate adhering to a rigid surface under the action of line forces and moments. (*a*) Natural configuration; and (*b*) equilibrium configuration.

figure 1*a*, point contact is only possible if the local curvature of the plate is more positive than that of the surface.

Interfacial adhesion and externally applied loads cause the contact to grow from a point to a circle of radius a, as shown in figure 1*b*. External loads can be applied as surface tractions (not shown in the figure) or line forces and moments at the edge of the plate. Moreover, the edge of the plate may be subject to kinematic constraints that prevent displacement and/or rotation. As with the geometry, the external loads and constraints are axisymmetric.

(*a*) *Kinematics*

Let the coordinate $s\in[0,L]$ denote the radial arclength of a point on the midplane, where L is the radial arclength of the plate. Next, define the rotation angle $\theta=\theta(s)$ of the undeformed midplane with respect to horizontal, as illustrated in figure 1*a*. At equilibrium, the rotation angle becomes $\theta+\phi$, where $\phi=\phi(s)$ is the change in angle induced by adhesion and external loads. At equilibrium, points may have also displaced tangentially by an amount $u_s=u_s(s)$. These deformations lead to elastic strains that also depend on the change in curvature $\phi'=\mathrm{d}\phi/\mathrm{d}s$ and tangential stretch $u_s'=\mathrm{d}u_s/\mathrm{d}s$.

Let $x=x(s)$ denote the radius of the plate at s. As shown in figure 1*b*, the radius becomes $x+u_x$ at equilibrium. Both $x(s)$ and $u_x(s)$ are evaluated by integrating over the interval $[0,s]$:

$$x=\int_0^s \cos\theta\,\mathrm{d}\hat{s} \quad \text{and} \quad u_x=\int_0^s \cos(\theta+\phi)\left(1+u_s'\right)\mathrm{d}\hat{s}-x. \tag{2.1}$$

Similarly, $y=y(s)$ and $u_y(s)$ denote the initial height and vertical displacement of a point at s,

$$y=\int_0^s \sin\theta\,\mathrm{d}\hat{s} \quad \text{and} \quad u_y=\int_0^s \sin(\theta+\phi)\left(1+u_s'\right)\mathrm{d}\hat{s}-y. \tag{2.2}$$

Because the undeformed plate is axisymmetric, it will not only have a curvature $\theta'+\phi'$ in the radial direction, but also a curvature κ in the hoop direction. At equilibrium, the curvature becomes $\kappa+\zeta$. From geometry it follows that

$$\kappa=\frac{\sin\theta}{x} \quad \text{and} \quad \zeta=\frac{\sin(\theta+\phi)}{x+u_x}-\kappa. \tag{2.3}$$

At equilibrium, the plate contacts the substrate over the interval $[0,a]$, where the radial arclength a of the contact zone is less than or equal to L. Although ϕ and u are continuous over the entire domain $[0,L]$, their derivatives ϕ' and u'_s will generally have a jump at $s=a$. Hence, it is convenient to define

$$\begin{aligned} \phi_\alpha &= \{\phi : s \in [0, a]\} \qquad \phi_\beta = \{\phi : s \in [a, L]\}, \\ u_\alpha &= \{u_s : s \in [0, a]\} \qquad u_\beta = \{u_s : s \in [a, L]\}. \end{aligned} \tag{2.4}$$

(b) Boundary conditions

Both θ and the surface profile of the substrate are prescribed and so it is straightforward to determine the angle change ϕ_α required for contact. If the interface is non-slip, then u_α may also be prescribed (e.g. $u_\alpha=0$). However, if the contact is frictionless then u_α is unknown and must be solved for. Regardless, the plate is subject to the following boundary conditions:

$$\phi_\alpha(a) = \phi_\beta(a) \quad \text{and} \quad u_\alpha(a) = u_\beta(a). \tag{2.5}$$

There may also be a kinematic constraint at the edge $s=L$ that leads to the boundary condition

$$\phi_\beta(L) = 0. \tag{2.6}$$

Lastly, there could be end constraints of the form

$$\int_0^L \{\sin(\theta+\phi)(1+u'_s) - \sin\theta\}\, \mathrm{d}s = c_1 \tag{2.7}$$

and

$$\int_0^L \{\cos(\theta+\phi)(1+u'_s) - \cos\theta\}\mathrm{d}s = c_2. \tag{2.8}$$

Equations (2.7) and (2.8) constrain the displacement of the edge of the plate in the vertical and radial directions respectively.

(c) Energy functional

The total potential energy of the system Π is composed of the elastic strain energy, the work of the applied forces, the virtual work of the reaction forces associated with the end contraints (2.7) and (2.8), and the energy of adhesion. For a von Kármán plate, the elastic energy density per unit area is

$$\psi = \frac{1}{2}D(\phi')^2 + \nu D\phi'\zeta + \frac{1}{2}D\zeta^2 + \frac{EH}{2(1-\nu^2)}\left\{(u'_s)^2 + 2\nu u'_s\frac{u_x}{x} + \left(\frac{u_x}{x}\right)^2\right\}, \tag{2.9}$$

where $D=EH^3/12(1-\nu^2)$ is the flexural rigidity; E is the elastic modulus; H is the plate thickness; and ν is Poisson's ratio. The work of the line forces and moment at $s=L$ are

$$2\pi x(L)F_x u_x(L), \qquad 2\pi x(L)F_y u_y(L) \quad \text{and} \quad 2\pi x(L)M\phi(L). \tag{2.10}$$

The virtual work associated with the end constraints are

$$2\pi x(L)\lambda_x u_x(L) \quad \text{and} \quad 2\pi x(L)\lambda_y u_y(L), \tag{2.11}$$

where the Lagrangian multipliers λ_x and λ_y represent the reaction forces necessary to maintain (2.7) and (2.8) (Lanczos 1970). The work of the surface tractions are obtained by integrating

$$\{(1+u_s')\cos(\theta+\phi)-\cos\theta\}\int_s^L 2\pi x t_x \,\mathrm{d}\hat{s} \tag{2.12}$$

and

$$\{(1+u_s')\sin(\theta+\phi)-\sin\theta\}\int_s^L 2\pi x t_y \,\mathrm{d}\hat{s}, \tag{2.13}$$

over the interval $[0,L]$. Lastly, γ denotes the energy of adhesion per unit area of contact.

The total potential energy is obtained by subtracting the work and adhesion energy from the elastic energy (Kendall 1971). This yields a functional of the form

$$\Pi = \int_0^a \mathcal{L}_\alpha \,\mathrm{d}s + \int_a^L \mathcal{L}_\beta \,\mathrm{d}s, \tag{2.14}$$

where the Lagrangian densities are

$$\mathcal{L}_\alpha = 2\pi x\psi_\alpha - 2\pi x(L)M\phi_\alpha' - 2\pi x\gamma - h_\alpha, \tag{2.15}$$

$$\mathcal{L}_\beta = 2\pi x\psi_\beta - 2\pi x(L)M\phi_\beta' - h_\beta. \tag{2.16}$$

Here, h_α and h_β are defined as $h(s,\phi_\alpha,u_\alpha,u_\alpha')$ and $h(s,\phi_\beta,u_\beta,u_\beta')$, respectively, where h is the integrand that corresponds to the sum of work of the applied line forces, reaction forces and surface tractions,

$$\begin{aligned} h &= 2\pi x(L)(1+u_s')\{(F_x+\lambda_x)\cos(\theta+\phi)+(F_y+\lambda_y)\sin(\theta+\phi)\} \\ &\quad +(1+u_s')\cos(\theta+\phi)\int_s^L 2\pi x t_x \,\mathrm{d}\hat{s} + (1+u_s')\sin(\theta+\phi)\int_s^L 2\pi x t_y \,\mathrm{d}\hat{s}. \end{aligned} \tag{2.17}$$

For convenience, the constant terms that do not contain ϕ, ϕ', u or u' are omitted from the energy functional.

3. Analysis and results

At equilibrium, the energy functional Π must be stationary with respect to kinematically admissible variations of the form

$$\left.\begin{aligned} &\phi_\alpha \to \phi_\alpha + \delta\phi_\alpha, \qquad \phi_\beta \to \phi_\beta + \delta\phi_\beta, \qquad u_\alpha \to u_\alpha + \delta u_\alpha, \\ &u_\beta \to u_\beta + \delta u_\beta \quad \text{and} \quad a \to a + \delta a. \end{aligned}\right\} \tag{3.1}$$

These conditions lead to the differential and boundary forms of the balance laws for both linear and angular momentum as well as a jump condition at the interface $s=a$. The main results are presented in §3*a*–*c*.

(a) Moment balance

Infinitesimal variations in ϕ_α and ϕ_β lead to a variation $\delta\Pi_\phi$ in the total potential energy of the system. At equilibrium, $\delta\Pi$ must vanish for any arbitrary functions $\delta\phi_\alpha$ and $\delta\phi_\beta$ that satisfy the kinematic boundary, i.e. the variations must be kinematically admissible. As shown in appendix A, $\delta\Pi=0$ if and only if the following conditions are satisfied:

$$\frac{\partial\mathcal{L}_\beta}{\partial\phi_\beta}-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\beta}{\partial\phi'_\beta}\right)=0, \tag{3.2}$$

$$\left(\frac{\partial\mathcal{L}_\beta}{\partial\phi'_\beta}\right)_{s=L}=0 \quad (\text{if } \phi_\beta(L) \text{ is not prescribed}). \tag{3.3}$$

Equations (3.2) and (3.3) represent the differential and boundary form of the moment balance, respectively.

(b) Force balance

Similarly, infinitesimal variations in u_α and u_β result in a variation $\delta\Pi_u$ that vanishes at equilibrium for arbitrary but kinematically admissible functions δu_α and δu_β. This implies (see appendix A)

$$\frac{\partial\mathcal{L}_\beta}{\partial u_\beta}-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\beta}{\partial u'_\beta}\right)=0, \tag{3.4}$$

$$\frac{\partial\mathcal{L}_\alpha}{\partial u_\alpha}-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\alpha}{\partial u'_\alpha}\right)=0 \quad (\text{if } u_\alpha \text{is not prescribed}), \tag{3.5}$$

$$\left(\frac{\partial\mathcal{L}_\alpha}{\partial u'_\alpha}\right)_{s=a}-\left(\frac{\partial\mathcal{L}_\beta}{\partial u'_\beta}\right)_{s=a}=0 \quad (\text{if } u_\alpha(a) \text{ is not prescribed}), \tag{3.6}$$

$$\left(\frac{\partial\mathcal{L}_\beta}{\partial u'_\beta}\right)_{s=L}=0 \quad (\text{if } u_\beta(L) \text{ is not prescribed}). \tag{3.7}$$

Equations (3.4), (3.5) and (3.6), (3.7) are equivalent to the differential and boundary forms of the linear force balance, respectively.

(c) Adhesive boundary condition

Lastly, varying the arclength a of the adhesive contact zone by an infinitesimal amount δa results in a variation of Π that has the form $\delta\Pi_a=(\mathrm{d}\Pi/\mathrm{d}a)\delta a$. At equilibrium, $\mathrm{d}\Pi/\mathrm{d}a$ must equal 0, which results in the following jump condition at $s=a$:

$$0=\mathcal{L}_\alpha(a)-\mathcal{L}_\beta(a)-\left(\frac{\partial\mathcal{L}_\beta}{\partial\phi'_\beta}\right)_{s=a}\{\phi'_\alpha(a)-\phi'_\beta(a)\}-\left(\frac{\partial\mathcal{L}_\beta}{\partial u'_\beta}\right)_{s=a}\{u'_\alpha(a)-u'_\beta(a)\}, \tag{3.8}$$

where the last term is dropped if u_α is not prescribed. Details of the derivation for (3.8) are presented in appendix B. In brief, the derivation involves Leibniz's integration rule and makes use of the boundary conditions and balance laws.

The adhesive boundary condition (3.8), which is derived using the calculus of variations, represents a special case of the second Weierstrass–Erdmann Corner Condition or the variable endpoint (free horizon) problem (Seifert 1991; Troutman 1996). In general, it can also regarded as an Eshelbian energy momentum balance or material (configurational) force balance at the edge ($s=a$) of the adhesive zone. In this respect, it may be possible to derive the adhesive boundary condition (3.8) using the J-integral method or configurational mechanics (Majidi 2007). In §4, we show that for a frictionless contact (such that u_α is not prescribed), the adhesive boundary condition can also be regarded as a discontinuity in the internal bending moment.

4. Moment discontinuity method

Suppose that the interface between the plate and substrate is frictionless. Since u_α is not prescribed, the last term in (3.8) must be dropped. Substituting the expressions for $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ into (3.8) and noting that x, u, u' and ϕ are continuous through $s=a$, the adhesive boundary condition becomes

$$-\gamma + \psi_\alpha(a) - \psi_\beta(a) - \left(\frac{\partial \psi_\beta}{\partial \phi'_\beta}\right)_{s=a} \{\phi'_\alpha(a) - \phi'_\beta(a)\} = 0. \tag{4.1}$$

Using the expression of ψ in (2.9), it follows that $\phi'_\alpha(a) - \phi'_\beta(a) = \pm\sqrt{2\gamma/D}$. The curvature may be represented in terms of the internal moment

$$M = \partial\psi/\partial\phi' = D\phi' + \nu D\zeta, \tag{4.2}$$

which implies $M_\alpha - M_\beta = \pm\sqrt{2D\gamma}$. According to the MDM, the right-hand side may be treated as an adhesion-induced singular moment $-\sqrt{2D\gamma}$ that is applied at $s=a$. The negative sign is chosen because the effect of adhesion is to produce an *applied* moment in the negative θ direction (figure 1*a*). If the rigid surface were on the top of the plate then the sign of this adhesion-induced moment would be positive. The jump condition can be expressed as

$$M_\beta = M_\alpha + M_o \quad \text{for } M_o = \sqrt{2D\gamma}, \tag{4.3}$$

where M_o is the *discontinuity* in the internal bending moment. Again if the rigid surface was above the plate then the sign of M_o would be reversed. An extension of this MDM into two plates can be found in Majidi & Wan (2009) in which the square root of the sum of the squares of the discontinuities in bending moments is equal to M_o.

5. Examples

By replacing adhesion with a line moment M_o at the contact boundary, the MDM allows plate adhesion problems to be studied entirely within the framework of classical plate theories. This is in contrast to the energy approach used in §3, the

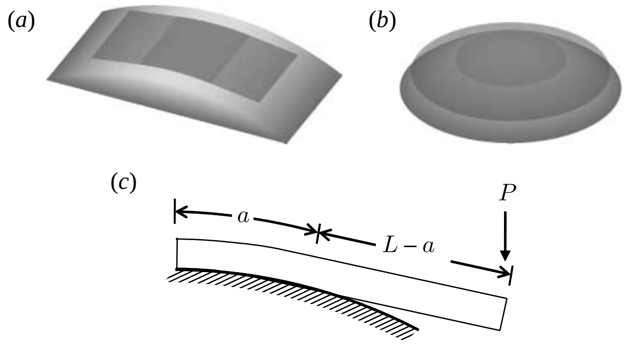

Figure 2. (*a*) Adhesion of a rectangular plate of length $2L$ to a rigid cylinder of radius R; (*b*) adhesion of a circular plate of radius L to a rigid sphere of radius R. (*c*) Both plates are subject to a line force $F_y = -P$ along their edge.

J-integral method, or the material (configurational) force balance, which all require insights and analyses that are outside the scope of Newtonian balance laws or moment-curvature constitutive models. For many one-dimensional and axisymmetric adhesion problems, MDM greatly simplifies the analysis and reduces derivation of the contact length a to only a few lines of algebra. This is demonstrated in the following examples.

(*a*) *Adhesion to a cylinder*

As illustrated in figure 2, a rectangular plate of length $2L$ is pressed onto a cylinder of radius R by a vertical line force $F_y = -P$ applied at its two ends. Both the adhesion and applied loads lead to a contact zone of length $2a \le 2L$ between the plate and cylinder. By symmetry only half of this configuration is analysed. At the edge of the contact zone, the plate has internal moments $M_\alpha = -D/R$ and $M_\beta = -P(L-a)$ just inside and outside of the zone, respectively. These internal moments must balance the singular moment $M_o = \sqrt{2D\gamma}$ induced at the edge:

$$M_\beta = M_\alpha + M_o \Rightarrow a = L - \frac{D}{PR} + \frac{1}{P}\sqrt{2D\gamma}. \tag{5.1}$$

It is noted that MDM was developed for an axisymmetric geometry. However, it is straightforward to show that this method also works for cylindrical bending. In the absence of a load P, the plate will either adhere completely ($a = L$) or not at all ($a = 0$) depending on whether the magnitude of M_α is less than or greater than M_o. It is straightforward, though significantly more tedious, to obtain a using an energy minimization argument.

(*b*) *Adhesion to a sphere*

Now suppose that a circular plate of radius L is pressed into a sphere of radius R. Referring to figure 2, the edge of the plate is subject to a vertical line force $F_y = -P$. According to MDM, the radius $a \le L$ of the contact zone must satisfy the moment balance $M_\beta = M_\alpha + M_o$. The internal moment inside but near the edge of the contact zone is $M_\alpha = -(1+\nu)D/R$ while the internal moment just outside of the contact zone is $M_\beta = D\{\phi'_\beta(a) + \nu\zeta\}$.

The end force P results in a line shear load $Q(s)=PL/s$. Therefore, from eqn (54) in Timoshenko & Woinowsky-Kreiger (1959, p. 53) and the boundary conditions $(2.5)_1$ and (3.3), it follows that $\phi_\beta=\phi_\beta(s)$ is the solution to

$$\phi''_\beta = -\frac{\phi'_\beta}{s} + \frac{\phi_\beta}{s^2} + \frac{PL}{Ds}, \qquad \phi_\beta(a) = -a/R \quad \text{and} \quad \phi'_\beta(L) + \nu\frac{\phi_\beta(L)}{L} = 0. \quad (5.2)$$

Alternatively, (5.2) may be derived by substituting the Lagrangian density obtained from §2*c* into the moment balance (3.2). Noting that the plate is initially flat ($\theta=0$), it follows from (2.1) and (2.3) that $x=s$, $u_x\approx 0$ and $\zeta\approx\phi_\beta/s$. Ignoring elastic stretching (u and u')

$$\mathcal{L}_\beta = \pi s D(\phi'_\beta)^2 + 2\pi\nu D\phi'_\beta\phi_\beta + \pi s D\left(\frac{\phi_\beta}{s}\right)^2 + 2\pi LP\phi_\beta. \quad (5.3)$$

An algebraic solution is obtained in MAPLE 12 and is used to evaluate $M_\beta = D\{\phi'_\beta(a) + \nu\phi_\beta(a)/a\}$. Substituting this into the moment balance $M_\beta=M_\alpha+M_o$:

$$-\frac{\nu D}{R} + \frac{(L^2-a^2)(2D-PRL) + 4\nu DLa + 2PRL^3\ln(a/L)}{2R(L^2+a^2)} \equiv -(1+\nu)\frac{D}{R} + \sqrt{2D\gamma}. \quad (5.4)$$

The contact radius at equilibrium is determined by solving the above equation for a.

The adhesion of a circular plate to a sphere was previously studied by Majidi & Fearing (2008). In their analysis, a bound on a was obtained by ignoring strain energy in the non-contacting portion (i.e. $\psi_\beta=0$) and performing an energy minimization. Their equilibrium condition (2.8) is equivalent to the condition $\gamma=\psi_\alpha(a)$ that results from (4.1) for $\psi_\beta=0$. Equation (5.4) represents a significant improvement on this result, since it allows for an applied force P and furnishes an explicit value for a at equilibrium rather than a bound.

(*c*) *Adhesion under a uniform surface traction*

Finally, consider the adhesion of a circular plate of radius L to a rigid flat surface under a uniform surface traction $t_y=-t$. As illustrated in figure 3, the plate is initially concave up with a slight spherical bow. The initial angle of rotation along the arclength is thus $\theta=s/R$, where R is the natural radius of curvature. For $R\gg L$ and assuming that the deflections are small compared to the thickness, the small-angle theorem may be employed and effects of the induced u' can be neglected. Here, the line shear load is $Q(s)=L^2/2s-s/2$ and so from eqn (54) in Timoshenko & Woinowsky-Kreiger (1959, p. 53), it follows that:[1]

$$\phi''_\beta + \frac{\phi'_\beta}{s} - \frac{\phi_\beta}{s^2} = \frac{t}{2D}\left(\frac{L^2}{s} - s\right), \quad (5.5)$$

which is subject to the boundary conditions $\phi_\beta(a)=-a/R$, $M_\beta(a)=-D(1+\nu)/R+M_o$ and $M_\beta(L)=0$. The solution of (5.5) subject to these boundary conditions gives

$$16(1+\nu) + \left\{\begin{array}{c} -(1-\nu)(a/L)^4 - 4\nu(a/L)^2 \\ +1+3\nu+4(1+\nu)\ln(a/L) \end{array}\right\}\frac{tRL^2}{D} = 8\sqrt{\frac{2\gamma L^2}{D}}, \quad (5.6)$$

[1] Equation (5.5) can also be obtained from the arguments in §2*c*. As in the previous example, we may approximate $x=s$, $u_x=0$ and $\zeta=\phi_\beta/s$.

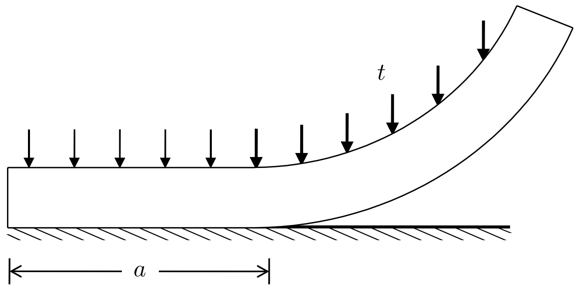

Figure 3. Adhesion of a spherically bowed circular plate of radius L to a flat rigid substrate and subject to a uniform surface traction t.

which relates the contact radius (a); the applied pressure (t); and work of adhesion (γ). This result is far simpler than that obtained by Pamp & Adams (2007), but gives the same numerical results.

6. Conclusion

In this paper, we have shown that the inclusion of adhesion, in a contact problem with an elastic plate, is equivalent to specifying a discontinuous internal bending moment at the contact region boundary. The magnitude of this moment discontinuity is equal to the square root of twice the product of the work of adhesion and the flexural rigidity of the plate. This formulation greatly reduces the algebraic complexity of solving adhesion problems with plates. Our moment discontinuity method can be considered a generalization of the well-known use of a continuous bending moment to solve plate contact problems without adhesion. The method has been implemented on some sample problems.

Appendix A. Derivation of moment and force balance

The first two variations in (3.1) lead to the following variation in the total potential energy of the system:

$$\delta\Pi_\phi = \int_0^a \left\{ \frac{\partial\mathcal{L}_\alpha}{\partial\phi_\alpha}\delta\phi_\alpha + \frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha}\delta\phi'_\alpha \right\} \mathrm{d}s + \int_a^L \left\{ \frac{\partial\mathcal{L}_\beta}{\partial\phi_\beta}\delta\phi_\beta + \frac{\partial\mathcal{L}_\beta}{\partial\phi'_\beta}\delta\phi'_\beta \right\} \mathrm{d}s. \tag{A 1}$$

By the chain rule (Lanczos 1970),

$$\int_0^a \frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha}\delta\phi'_\alpha \,\mathrm{d}s = \int_0^a \left\{ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha}\delta\phi_\alpha \right) - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha} \right)\delta\phi_\alpha \right\} \mathrm{d}s$$
$$= \left[\frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha}\delta\phi_\alpha \right]_0^a - \int_0^a \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha} \right)\delta\phi_\alpha \,\mathrm{d}s. \tag{A 2}$$

Following the same argument for the other integral in (A 1), leads to the result:

$$\delta\Pi_\phi = \left[\frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha}\delta\phi_\alpha\right]_0^a + \int_0^a \left\{\frac{\partial\mathcal{L}_\alpha}{\partial\phi_\alpha} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\alpha}{\partial\phi'_\alpha}\right)\right\}\delta\phi_\alpha\,\mathrm{d}s + \left[\frac{\partial\mathcal{L}_\beta}{\partial\phi'_\beta}\delta\phi_\beta\right]_a^L + \int_a^L \left\{\frac{\partial\mathcal{L}_\beta}{\partial\phi_\beta} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\beta}{\partial\phi'_\beta}\right)\right\}\delta\phi_\beta\,\mathrm{d}s. \tag{A 3}$$

According to the boundary condition that ϕ_α is prescribed, $\delta\phi_\alpha$ must equal 0 for all $s\in[0,a]$. Moreover, $\delta\phi_\beta(a)=0$ since $\phi_\beta(a)$ is constrained to be equal to the fixed value $\phi_\alpha(a)$. Also, if ϕ_β is prescribed at $s=L$ then $\delta\phi_\beta(L)=0$ as well. For s between a and L, $\delta\phi_\beta$ is arbitrary and so $\delta\Pi$ vanishes only if

$$\frac{\partial\mathcal{L}_\beta}{\partial\phi_\beta} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\beta}{\partial\phi'_\beta}\right) = 0. \tag{A 4}$$

If $\phi_\beta(L)$ is not prescribed then $\delta\phi_\beta(L)$ is also arbitrary and so the condition $\delta\Pi=0$ also requires $(\partial\mathcal{L}_\beta/\partial\phi'_\beta)_{s=L}=0$.

Using a similar argument, it can be shown that variations of the third and fourth terms in (3.1) yield

$$\delta\Pi_u = \left[\frac{\partial\mathcal{L}_\alpha}{\partial u'_\alpha}\delta u_\alpha\right]_0^a + \int_0^a \left\{\frac{\partial\mathcal{L}_\alpha}{\partial u_\alpha} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\alpha}{\partial u'_\alpha}\right)\right\}\delta u_\alpha\,\mathrm{d}s + \left[\frac{\partial\mathcal{L}_\beta}{\partial u'_\beta}\delta u_\beta\right]_a^L + \int_a^L \left\{\frac{\partial\mathcal{L}_\beta}{\partial u_\beta} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\beta}{\partial u'_\beta}\right)\right\}\delta u_\beta\,\mathrm{d}s. \tag{A 5}$$

The condition $\delta\Pi_u=0$ requires that

$$\frac{\partial\mathcal{L}_\beta}{\partial u_\beta} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\beta}{\partial u'_\beta}\right). \tag{A 6}$$

Also, if u_α is not prescribed then $\delta\Pi_u=0$ only holds if

$$\frac{\partial\mathcal{L}_\alpha}{\partial u_\alpha} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}_\alpha}{\partial u'_\alpha}\right), \tag{A 7}$$

and $(\partial\mathcal{L}_\alpha/\partial u'_\alpha)_{s=a}=(\partial\mathcal{L}_\beta/\partial u'_\beta)_{s=a}$. Also, if $u_\beta(L)$ is not prescribed, then equilibrium requires $(\partial\mathcal{L}_\beta/\partial u'_\beta)_{s=L}=0$.

Appendix B. Derivation of adhesive boundary condition

The variation of the fifth term in (3.1) corresponds to

$$\delta\Pi_a = \frac{\mathrm{d}\Pi_a}{\mathrm{d}a}\delta a = \mathcal{L}_\alpha(a)\delta a + \int_0^a \left\{ \begin{array}{c} \dfrac{\partial \mathcal{L}_\alpha}{\partial a} + \dfrac{\partial \mathcal{L}_\alpha}{\partial \phi_\alpha}\dfrac{\mathrm{d}\phi_\alpha}{\mathrm{d}a} + \dfrac{\mathrm{d}\mathcal{L}_\alpha}{\mathrm{d}\phi'_\alpha}\dfrac{\mathrm{d}\phi'_\alpha}{\mathrm{d}a} \\ + \dfrac{\partial \mathcal{L}_\alpha}{\partial u_\alpha}\dfrac{\mathrm{d}u_\alpha}{\mathrm{d}a} + \dfrac{\mathrm{d}\mathcal{L}_\alpha}{\mathrm{d}u'_\alpha}\dfrac{\mathrm{d}u'_\alpha}{\mathrm{d}a} \end{array} \right\} \delta a\, \mathrm{d}s$$

$$- \mathcal{L}_\beta(a)\delta a + \int_a^L \left\{ \begin{array}{c} \dfrac{\partial \mathcal{L}_\beta}{\partial a} + \dfrac{\partial \mathcal{L}_\beta}{\partial \phi_\beta}\dfrac{\mathrm{d}\phi_\beta}{\mathrm{d}a} + \dfrac{\mathrm{d}\mathcal{L}_\beta}{\mathrm{d}\phi'_\beta}\dfrac{\mathrm{d}\phi'_\beta}{\mathrm{d}a} \\ + \dfrac{\partial \mathcal{L}_\beta}{\partial u_\beta}\dfrac{\mathrm{d}u_\beta}{\mathrm{d}a} + \dfrac{\mathrm{d}\mathcal{L}_\beta}{\mathrm{d}u'_\beta}\dfrac{\mathrm{d}u'_\beta}{\mathrm{d}a} \end{array} \right\} \delta a\, \mathrm{d}s$$

$$= 0. \tag{B 1}$$

By the balance law (3.2) and the chain rule,

$$\int_a^L \left\{ \frac{\partial \mathcal{L}_\beta}{\partial \phi_\beta}\frac{\mathrm{d}\phi_\beta}{\mathrm{d}a} + \frac{\mathrm{d}\mathcal{L}_\beta}{\mathrm{d}\phi'_\beta}\frac{\mathrm{d}\phi'_\beta}{\mathrm{d}a} \right\}\mathrm{d}s = \int_a^L \left\{ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathcal{L}_\beta}{\partial \phi'_\beta}\right)\frac{\mathrm{d}\phi_\beta}{\mathrm{d}a} + \frac{\mathrm{d}\mathcal{L}_\beta}{\mathrm{d}\phi'_\beta}\frac{\mathrm{d}\phi'_\beta}{\mathrm{d}a} \right\}\mathrm{d}s$$

$$= \int_a^L \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathcal{L}_\beta}{\partial \phi'_\beta}\frac{\mathrm{d}\phi_\beta}{\mathrm{d}a}\right)\mathrm{d}s$$

$$= \left(\frac{\partial \mathcal{L}_\beta}{\partial \phi'_\beta}\frac{\mathrm{d}\phi_\beta}{\mathrm{d}a}\right)_{s=L} - \left(\frac{\partial \mathcal{L}_\beta}{\partial \phi'_\beta}\frac{\mathrm{d}\phi_\beta}{\mathrm{d}a}\right)_{s=a}. \tag{B 2}$$

The variation $\delta\Pi_a$ is simplified by following the same argument for the other integrals and noting that $\mathrm{d}\phi_\alpha/\mathrm{d}a=0$ and $(\mathrm{d}u_\alpha/\mathrm{d}a)_{s=0}=0$. Also, if $u_\beta(L)$ and $\phi_\beta(L)$ are prescribed then $\mathrm{d}u_\beta/\mathrm{d}a=\mathrm{d}\phi_\beta/\mathrm{d}a=0$ at $s=L$. Otherwise, it follows from (3.3) and (3.7) that $\partial\mathcal{L}_\beta/\partial u'_\beta=\partial\mathcal{L}_\beta/\partial\phi'_\beta=0$. In either case,

$$\left(\frac{\partial \mathcal{L}_\beta}{\partial u'_\beta}\frac{\mathrm{d}u_\beta}{\mathrm{d}a}\right)_{s=L} = \left(\frac{\partial \mathcal{L}_\beta}{\partial \phi'_\beta}\frac{\mathrm{d}\phi_\beta}{\mathrm{d}a}\right)_{s=L} = 0. \tag{B 3}$$

Hence,

$$\frac{\mathrm{d}\Pi_a}{\mathrm{d}a} = \mathcal{L}_\alpha(a) - \mathcal{L}_\beta(a) - \left(\frac{\partial \mathcal{L}_\beta}{\partial \phi'_\beta}\frac{\mathrm{d}\phi_\beta}{\mathrm{d}a}\right)_{s=a} - \left(\frac{\partial \mathcal{L}_\beta}{\partial u'_\beta}\frac{\mathrm{d}u_\beta}{\mathrm{d}a}\right)_{s=a} + \left(\frac{\partial \mathcal{L}_\alpha}{\partial u'_\alpha}\frac{\mathrm{d}u_\alpha}{\mathrm{d}a}\right)_{s=a}. \tag{B 4}$$

If u_α is not prescribed, then $u=u(s)$ is smooth at $s=a$ since both u_α and u_β are solutions to the same balance equations and subject to the boundary conditions $u_\alpha(a)=u_\beta(a)$ and (3.6). Hence $(\mathrm{d}u_\alpha/\mathrm{d}a)_{s=a}=(\mathrm{d}u_\beta/\mathrm{d}a)_{s=a}$ and so in light of (3.6) this implies that the last two terms of (B 4) cancel each other out. However, if u_α is prescribed then $(\mathrm{d}u_\alpha/\mathrm{d}a)_{s=a}=0$ and the second to last term will remain.

In order to determine $\mathrm{d}\phi_\beta/\mathrm{d}a$ and $\mathrm{d}u_\beta/\mathrm{d}a$ at $s=a$, consider a variation of the form $a=a^*+\delta a$, where a^* is the arclength of the contact zone at equilibrium and δa represents an infinitesimal variation from a^* (Seifert 1991). This results in variations of the form $\phi_\beta=\phi_\beta^*+\delta\phi_\beta$ and $u_\beta=u_\beta^*+\delta u_\beta$. According to the boundary conditions (and noting that the variations must be kinematically admissible), $\phi_\beta^*(a^*)=\phi_\alpha(a^*)$, $\phi_\beta(a)=\phi_\alpha(a)$, $u_\beta^*(a^*)=u_\alpha(a^*)$ and $u_\beta(a)=u_\alpha(a)$, where both ϕ_α and u_α are prescribed. Moreover, according to the fundamental theorem of calculus, for any function $\chi=\chi(s)$, $\chi(a)=\chi(a^*)+\delta a\chi'(a^*)+\mathcal{O}(\delta a^2)$. Therefore,

$$\phi_\beta(a)=\phi_\beta(a^*)+\delta a\phi_\beta'(a)=\phi_\beta^*(a^*)+\delta\phi_\beta(a)+\delta a\phi_\beta'(a), \tag{B 5}$$

which, according to the boundary condition $\phi_\beta(a)=\phi_\alpha(a)$, is equivalent to

$$\phi_\alpha(a)=\phi_\alpha(a^*)+\delta a\phi_\alpha'(a). \tag{B 6}$$

Noting that $\phi_\alpha(a^*)=\phi_\beta^*(a^*)$ and solving for $\delta\phi_\beta(a)/\delta a\equiv(\mathrm{d}\phi_\beta/\mathrm{d}a)_{s=a}$ implies that $(\mathrm{d}\phi_\beta/\mathrm{d}a)_{s=a}=\phi_\alpha'(a)-\phi_\beta'(a)$. Similarly, $(\mathrm{d}u_\beta/\mathrm{d}a)_{s=a}=u_\alpha'(a)-u_\beta'(a)$. Hence, the condition $\delta\Pi_a=0$ implies

$$0=\mathcal{L}_\alpha(a)-\mathcal{L}_\beta(a)-\left(\frac{\partial\mathcal{L}_\beta}{\partial\phi_\beta'}\right)_{s=a}\{\phi_\alpha'(a)-\phi_\beta'(a)\}-\left(\frac{\partial\mathcal{L}_\beta}{\partial u_\beta'}\right)_{s=a}\{u_\alpha'(a)-u_\beta'(a)\}. \tag{B 7}$$

If u_α is not prescribed then the last term must be dropped.

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