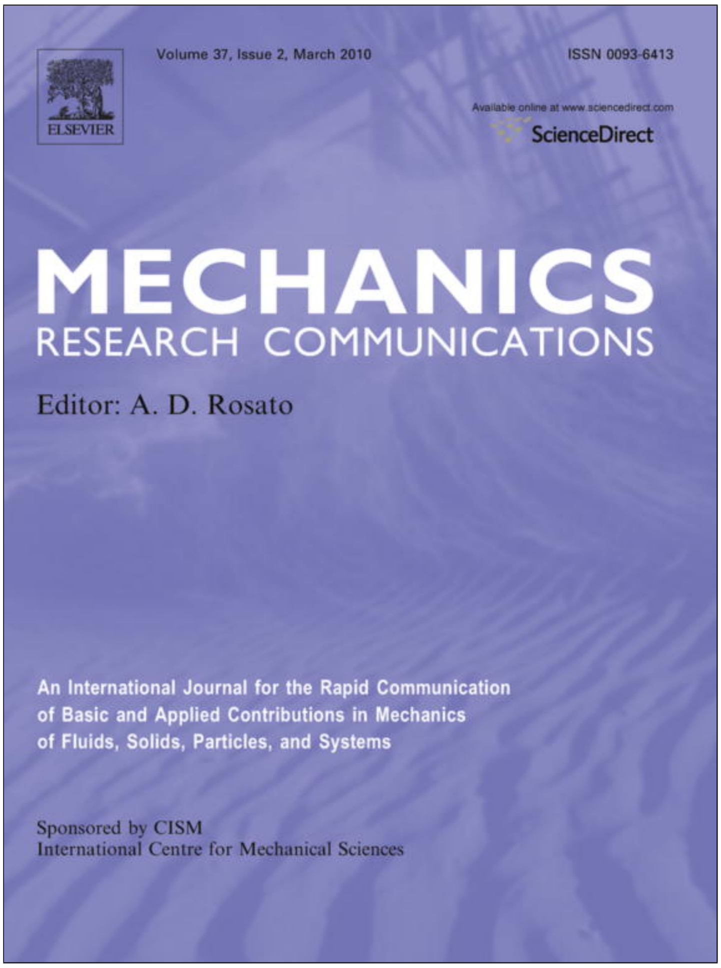

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Adhesion and delamination boundary conditions for elastic plates with arbitrary contact shape

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ABSTRACT

The adhesion of two heterogeneous, thin-walled structures is shown to be controlled by a boundary condition that balances mechanical energies with the work of adhesion at the edge of the contact zone. This boundary condition is well-known in fracture mechanics but is here rederived with plate theory and represented in a form that is easy to use. This formulation is applicable either to problems in which the contact area is *a priori* unknown, or in problems in which the bonded area is predefined and it is the onset of debonding that is of interest. The simplified boundary condition is shown to be very useful and simple to use in both cases, but particularly in the latter class of problems. In the case of one-dimensional or axisymmetric problems where one of the bodies is rigid, this representation is equivalent to the Moment-Discontinuity-Method (MDM) introduced by Pamp and Adams and Majidi and Adams.

1. Introduction

Adhesion of shells, plates, and other thin-walled structures governs the functionality of many natural systems, fabrication methods, and emerging technologies, from biological cell adhesion (Seifert and Lipowsky, 1990; Wan and Liu, 2001) to semiconductor wafer bonding (Tong and Gosele, 1994; Pamp and Adams, 2007; Turner and Spearing, 2002) and flexible electronics (Vella et al., 2009). Progress in these fields is aided by theories, computational simulations, and design rules that are often based on a contact potential representation of adhesion. This approach was formalized in Kendall's theory of adhesion (Kendall, 1971) and has its origins in Griffith's theory of fracture (Griffith, 1921). Both theories require the minimization of potential energy, which is composed not only of surface and interfacial energy but also the elastic strain energy of the deforming bodies and the work of external loads.

Even in the case of one-dimensional and axisymmetric systems, performing such an energy minimization can lead to extremely lengthy and tedious calculations (Pamp and Adams, 2007). The resulting internal forces and bending moment undergo a discontinuity across the edge of the contact area Ω_c (Pamp and Adams, 2007; Turner and Spearing, 2002; Majidi and Adams, 2009; Springman and Bassani, 2008), which are induced by the adhesive interactions. In Majidi and Adams (2009) it was shown that these jumps correspond to a boundary condition at the edge of Ω_c that is obtained by balancing the interfacial work of adhesion (γ) with the strain energy release rate ($\mathscr{G}$).

Alternatively the boundary condition ($\mathscr{G} = \gamma$) can be determined using methods from fracture mechanics such as the *J*-integral (Glassmaker and Hui, 2004; Rice, 1968) and the stress intensity factor (Suo and Hutchinson, 1990). Such methods, however, go beyond the scope of conventional plate or shell theories and require the evaluation of internal stress and strain fields. Nonetheless, the theory presented here is based on the same hypotheses of linear elasticity and Griffith's contact potential that furnish linear elastic fracture mechanics (LEFM).

Here, the boundary condition for adhesion between plates, membranes, and shells is derived and expressed entirely within the context of conventional shell theory. In contrast to some of the previous solutions (Seifert and Lipowsky, 1990; Majidi and Adams, 2009; Glassmaker and Hui, 2004), the current result is valid for structures that share an irregular, non-axisymmetric contact area and that are dissimilar in shape and elastic rigidity. In addition to flexure and stretching, this result also includes transverse and in-plane shear deformation. Hence, the result is a generalization to the solution obtained in Suo and Hutchinson (1990) for the debonding of a laminated composite.

This formulation is applied to examples that represent two distinct classes of plate adhesion problems. In one example, the shape of the contact area Ω_c is not known *a priori* and must be determined by solving a boundary value problem. In the other example, which concerns delamination in a MEMS structure, Ω_c is predefined and it is the onset of debonding that is of interest. In both cases, the boundary condition is quite useful and greatly simplifies

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the complexity of the analysis. This is particularly true for the latter class of problems in which the onset of debonding must be determined.

2. System description

The adhering thin-walled structures are treated as two-dimensional surfaces that adhere over a domain Ω_c. As illustrated in Fig. 1a, the non-contacting portions of the two surfaces are denoted by Ω_a and Ω_b. The surfaces intersect along the interface $\mathscr{B} = \Omega_c \cap \Omega_a = \Omega_c \cap \Omega_b$. The space curve $\mathscr{B}$ and surfaces Ω_a, Ω_b, and Ω_c are all embedded in the Euclidean space $\mathbb{R}^3$.

The reference kinematics are defined with respect to the final configuration of the system at equilibrium. Points $\mathbf{X} \in \Omega_c$ are uniquely identified using a curvilinear coordinate system $\{\xi, \zeta\}$ that is fitted to the interface $\mathscr{B}$. As shown in Fig. 1a, the coordinate curves for ξ and ζ are normal and tangent to the boundary $\mathscr{B}$, respectively. The arclength v is defined such that $\mathscr{B} = \{\mathbf{X} : \xi = v\}$ and, depending on the coordinate system selected, may or may not be a function of ζ. The unit normal vector $\mathbf{n}$ is thus equivalent to the normalized director $\partial\mathbf{X}/\partial\xi$ evaluated along $\mathbf{X} \in \mathscr{B}$. Similarly, the unit tangent vector $\mathbf{t}$ equals the normalized director $\partial\mathbf{X}/\partial\zeta$.

Referring to the inset in Fig. 1a, the curvature κ is defined as $\kappa = d\psi/d\xi$, where ψ is the rotation in the $\mathbf{n}$ direction of a line initially perpendicular to the midplane (Marguerre and Woernle, 1969). It is noted that in the absence of transverse shear deformation, κ would represent the curvature of the neutral surface. Even in the natural (non-contacting, undeformed) configuration, the plates may have intrinsic curvature κ^0, as in the case of a cylindrical shell. In addition to bending strains, the plates are also subject to a membrane strain μ along the $\mathbf{n}$ direction. Membrane and bending strains will also occur in the $\mathbf{t}$ direction, although these values drop out when deriving the boundary condition. Also, for the sake of clarity, transverse and in-plane shear deformations are ignored in the derivation but are later accounted for with correction terms that are added to the final result.

Deformation is limited to elastic modes of flexure (bending) and planar stretching. As in conventional plate and shell theories, these deformations induce line forces and bending moments. These internal loads are governed by the elastic rigidities of flexure $D = EH^3/12(1-\nu^2)$ and stretching $K = EH/(1-\nu^2)$, where E is the elastic modulus and ν is Poisson's ratio (Marguerre and Woernle, 1969). In general, these elastic rigidities will be different for the two plates and will be denoted by D_α and K_α. Here, as in the remainder of the paper, $\alpha \in \{a, b\}$ and the indices 'a', 'b', and 'c' denote the values on surfaces Ω_a, Ω_b, and Ω_c, respectively.

Within the contact zone Ω_c, the plates are bonded together and behave like a composite. Hence, bending will increase the average axial strain in each plate by an amount μ_c^α:

$$\mu_c^a = (\kappa_a^0 - \kappa_c)\left(\frac{H_a}{2} + H_b - h\right) \quad \text{and} \quad \mu_c^b = (\kappa_c - \kappa_b^0)\left(h - \frac{H_b}{2}\right). \tag{1}$$

Here, h denotes the distance of the neutral surface from the non-contacting side of the plate corresponding to Ω_b:

$$h = \frac{K_aH_a + K_bH_b + 2K_aH_b}{2(K_a + K_b)}. \tag{2}$$

Eqs. (1) and (2) are adapted from Eqs. (1.1) and (AIII.1) in Suo and Hutchinson (1990). These expressions arise because the two plates act together as a single composite plate in Ω_c.

Variations in the elastic deformation of the plates are balanced by the mechanical work of external forces. In general, these may include body forces and external tractions that are applied on the surface or along the edges. As the plates deform, these tractions will displace and exert mechanical work on the system. The plates are also subjected to interfacial forces associated with the adhesive bonds formed inside the contact zone Ω_c. As the plates come out of contact, these interfacial forces perform mechanical work that is approximately equal to the *work of adhesion* γ. In reality, γ corresponds to the work necessary to displace two surfaces of unit area from intimate contact to infinite separation. By replacing the adhesive forces with the contact potential γ we are implicitly assuming that the range of the interfacial forces is negligible compared to the interfacial gap between the non-contacting surfaces. Such an approximation is reasonable for micron and millimeter scale systems, where the interfacial forces are due to short-range chemical or van der Waals interactions.

3. Analysis

The total potential energy of the system Π is composed of the potential energy associated with the mechanical work of external forces (including adhesion, γ) as well as the strain energy ψ associated with elastic deformation. At equilibrium, Π must be stationary with respect to arbitrary, infinitesimal, and kinematically admissible variations in the elastic deformation of the surfaces. In contrast to conventional plate and shell theories, the current system contains a *free interface* $\mathscr{B} = \{\mathbf{X} : \xi = v\}$ that is also subject to variation. Specifically, Π must be stationary with respect to variations of the form $\mathscr{B} \to \mathscr{B}' = \{\mathbf{X} : \xi = v - \epsilon\phi\}$, where $\phi = \phi(\zeta)$ is arbitrary and $\epsilon > 0$ is infinitesimally small.

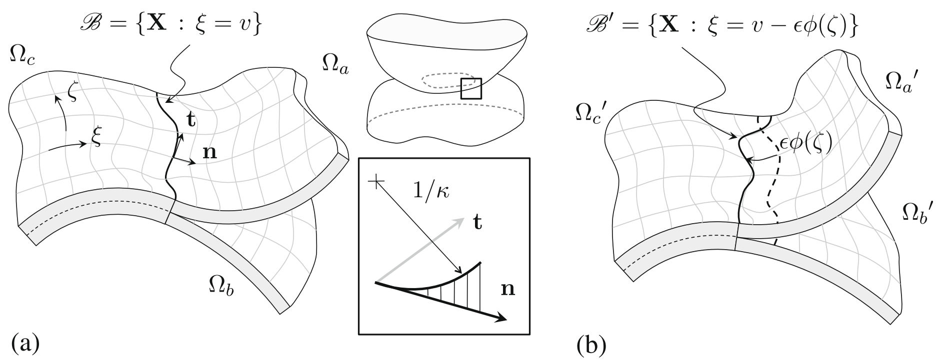

Fig. 1. (a) Thin-walled structures adhere over the domain Ω_c; $\{\xi, \zeta\}$ represents a boundary-fitted curvilinear coordinate system, where the coordinate curves for ξ and ζ are normal and tangent to the boundary $\mathscr{B}$, respectively. As shown in the inset, the surface has a curvature κ in the $\mathbf{n}$ direction. (b) At equilibrium, the total potential energy of the system Π must be stationary with respect to variations of the form $\mathscr{B} \to \mathscr{B}'$. This is identified with displacing the adhesion front $\mathscr{B}$ by an amount $\delta v = |\epsilon\phi|$, where the function $\phi = \phi(\zeta)$ is arbitrary and ϵ is infinitesimally small.

Physically, the variation $\mathscr{B} \to \mathscr{B}'$ corresponds to exchanging material line segments δv of length $\epsilon|\phi|$ between the contact zone Ω_c and the non-contacting portions Ω_a and Ω_b. This process, illustrated in Fig. 1b, leads to a variation in the potential energy of the form

$$\delta\Pi = \int_{\mathscr{B}} \{\delta\psi + \gamma\epsilon\phi + \delta W\} d\zeta. \tag{3}$$

Here, $\delta\psi$ is the change in the elastic strain energy stored in the segment δv and δW is the mechanical work performed by that segment on Ω_a, Ω_b, and the remaining portion of Ω_c.

At equilibrium, the variation $\delta\Pi$ must vanish for arbitrary $\phi = \phi(\zeta)$. The first step in evaluating $\delta\Pi$ is to determine the variation in strain energy $\delta\psi$. Across the boundary $\mathscr{B}$ of the contact zone, the bending curvature κ and membrane strain μ (both in the $\mathbf{n}$ direction) will undergo a jump. In contrast, bending curvature and stretch in the tangent $\mathbf{t}$ direction remain fixed. Therefore, as the segment δv delaminates, its elastic strain energy changes by an amount

$$\delta\psi = \frac{1}{2}\left\{D_\alpha\left[(\kappa_\alpha - \kappa_\alpha^0)^2 - (\kappa_c - \kappa_\alpha^0)^2\right] + K_\alpha\left[\mu_\alpha^2 - (\mu_c + \mu_c^\alpha)^2\right]\right\}\epsilon\phi, \tag{4}$$

which must be summed over the indices $\alpha \in \{a, b\}$.

The next step is to determine the mechanical work δW performed by the segment δv on the rest of the system. This is equivalent to the variation in the potential energy of the rest of the system and is equal and opposite to the work performed on δv. Therefore, δW has the form

$$\delta W = -\left\{D_\alpha(\kappa_\alpha - \kappa_\alpha^0)(\kappa_\alpha - \kappa_c) + K_\alpha\mu_\alpha\left[\mu_\alpha - (\mu_c + \mu_c^\alpha)\right]\right\}\epsilon\phi. \tag{5}$$

Combining these equations and noting that the integrand in (3) must vanish for arbitrary ϕ, it follows that

$$\gamma = \frac{1}{2}\left\{D_\alpha(\kappa_\alpha - \kappa_c)^2 + K_\alpha\left[\mu_\alpha - (\mu_c + \mu_c^\alpha)\right]^2\right\}. \tag{6}$$

This result is equivalent to the Griffith balance $\mathscr{G} = \gamma$ used in LEFM and is consistent with the boundary condition for a delaminating composite presented in Suo and Hutchinson (1990). Eq. (6) might also be derived using the J-integral method (Rice, 1968), although this requires the introduction of second-order tensors that are beyond the scope of the present theory. Lastly, this result is a limit of the boundary condition obtained from cohesive zone models, such as the one presented in Pamp and Adams (2007).

In the special case when one of the surfaces, say Ω_b, is rigid, the boundary condition becomes

$$\gamma = \frac{1}{2}\left\{D_a(\kappa_a - \kappa_c)^2 + K_a(\mu_a - \mu_c)^2\right\}. \tag{7}$$

This corresponds to the jump condition derived in Majidi and Adams Majidi and Adams (2009) except that here we have not assumed that the plates are axisymmetric.

4. Solution with shear deformation

The boundary condition (6) is applicable to systems such as thin plates and membranes for which shear deformation can be neglected. In general, however, elastic shear strains may also contribute to the potential energy of the system. Define $\mathbf{z} = \mathbf{n} \times \mathbf{t}$ and let φ and η denote the transverse and in-plane shear strains in the $\mathbf{n}$–$\mathbf{z}$ and $\mathbf{n}$–$\mathbf{t}$ planes, respectively. These correspond to the transverse shear rigidity $S^t = (5/6)GH$ and in-plane shear rigidity $S^p = GH$, where $G = E/2(1+\nu)$ is the shear modulus (Marguerre and Woernle, 1969). Here, the shear correction factor 5/6 is obtained by assuming a parabolic distribution of shear stress through the thickness of the plate (Reissner, 1945). In general, these rigidities will be different for the two plates and are denoted by S_α^t and S_α^p, where again $\alpha \in \{a, b\}$.

The contributions of shear deformation are handled in much the same way as the strain μ. At the boundary $\mathscr{B}$, both φ and η may undergo a jump and so the boundary condition becomes

$$\gamma = \frac{1}{2}\Big\{D_\alpha(\kappa_\alpha - \kappa_c)^2 + K_\alpha\left[\mu_\alpha - (\mu_c + \mu_c^\alpha)\right]^2 + S_\alpha^p(\eta_\alpha - \eta_c)^2 + S_\alpha^t\varphi_\alpha^2 - 2S_\alpha^t\varphi_\alpha\varphi_c + S_c^t\varphi_c^2\Big\}, \tag{8}$$

where S_c^t is the rigidity of the bonded plates. It is important to note that because of the transverse shear strain φ, $\kappa = d\psi/d\xi$ is no longer equivalent to the curvature of the neutral surface. Eq. (8) represents a generalization of the boundary condition presented in Suo and Hutchinson (1990).

5. Discussion and examples

Stationary principles and mechanics are used to derive the boundary condition (8) for adhesion between elastic plates with dissimilar geometry and stiffness. This boundary condition is applicable at the edge of the contact zone and suggests an adhesion induced jump in elastic bending curvature, stretch, and shear strain. It is valid for all plate geometries and loading conditions and allows such problems to be studied with conventional plate theory. Moreover, adding Eq. (8) to the governing balance laws eliminates the need to rederive the equilibrium condition $\delta\Pi = 0$ for each system that is studied.

There are two classes of adhesion problems in which this method can be expected to be useful. The first type of problem is one in which a plate makes contact with either another plate or with a rigid surface in such a manner that the contact zone need not be circular nor uniform across its width. In these situations the aim is to determine the contact area, after which displacements and/or stresses can be determined as needed. The other class of problems is characterized by a plate in bonded contact over an arbitrarily shaped predefined area with either another plate or with a rigid surface. In those cases the question to be answered is "Will debonding occur and, if so, where will debonding initiate?" Two examples will be given – one for each type of problem.

5.1. Compression induced delamination

Consider a long flat plate of flexural rigidity D lying on a rigid surface and under uniform compression in the x-direction with force per unit length Q. The possibility of separation from the surface non-uniformly in the y-direction is investigated. The simpler case of uniformity in the y-direction is possible but not considered here. The homogeneous solution of the partial differential equation for an elastic plate compressed in the x-direction is given by Jahanshahi and Dundurs (1964) as

$$w(r,\theta) = \sin\lambda x \sum_{k=1,3,5,\ldots}^{2N-1} A_k J_k(\lambda r)\cos k\theta + \cos\lambda x \sum_{k=0,2,4,\ldots}^{2N-2} A_k J_k(\lambda r)\cos k\theta, \tag{9}$$

where $w(r,\theta)$ is the transverse deflection, $\lambda = \sqrt{Q/4D}$, $J_k(:)$ are the Bessel functions of the first kind, and the series has been truncated to a total of $2N$ terms. It is noted that the more general solution of the partial differential equation contains Bessel functions of the second kind that lead to unbounded behavior at the origin and are thus omitted. Also symmetry about both the x- and y-directions has been used in order to eliminate the even terms of A_k and the odd terms of B_k.

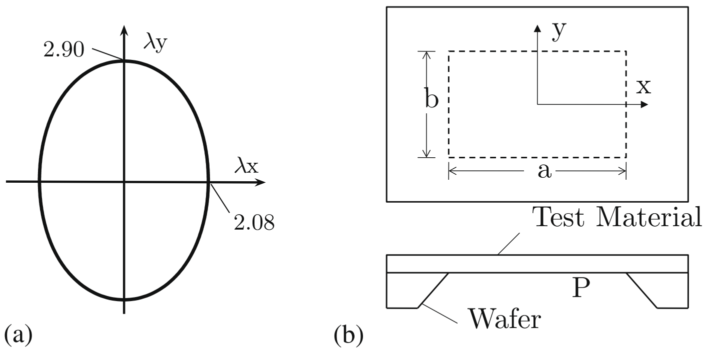

Fig. 2. (a) Contact boundary $R(\theta)$ of the debonded zone under a compressive stress in the x-direction. (b) Illustration of the MEMS blister test.

The solution procedure is to guess a contact boundary $R = R(\theta)$ for N uniformly spaced θ_i in $[0, \pi/2]$ and using $w(R_i, \theta_i) = 0$ and $M_{nn}(R_i, \theta_i) = \sqrt{2D\gamma}$ for $i = 1, 2, \ldots N$ to solve the resulting linear algebraic equations. The condition $M_{nn} = \sqrt{2D\gamma}$ is obtained from (7) by letting $D_a = D$, assuming a frictionless contact (such that $\mu_a = \mu_c$), noting that $\kappa_c = 0$, and using the constitutive law to replace κ_a with M_{nn}/D. In general the guess for $R_i = R(\theta_i)$ will not solve the zero slope boundary condition

$$\frac{\partial w}{\partial r}(R_i, \theta_i) = 0, \quad i = 1, 2, \ldots N. \tag{10}$$

It is noted that because the displacement is zero along the boundary, the requirement that the slope normal to the boundary vanishes becomes equivalent to (10). Thus (10) represents N nonlinear algebraic equations for R_i which can be solved by standard methods. The resulting shape of the separation zone is shown in Fig. 2a.

5.2. MEMS blister test

As an example of the second class of problems consider the blister test often used to test materials in MEMS. A layer of the test material is deposited on a wafer and then a rectangular (usually square) window is etched from the back-side of the wafer leaving a rectangular portion of the thin test material exposed, as shown in Fig. 2b. A pressure is then exerted on the material by a gas and the plate deflection is measured, thus allowing material properties, such as Young's modulus, to be determined. This test is not used to measure adhesion; the assumption is that there is sufficient adhesion to maintain a clamped condition all along the edges. In this problem our goal is to determine the maximum applied pressure for which the bond will remain intact. By applying Eq. (6) it is seen that debonding will initiate at a point along the boundary for which the internal bending moment M_{nn} along an edge becomes equal to $\sqrt{2D\gamma}$.

The solution for this problem is extremely simple. The maximum moment in the normal direction, along the boundary of a rectangular plate clamped along its edges and subjected to a uniform pressure, occurs at the midpoints of the longer edge (Timoshenko and Woinowsky-Krieger, 1959). For a square plate this maximum moment is given by $M_{nn}(\pm a/2, 0) = 0.0513\ pa^2$. Using this result gives a critical pressure (p_c) of $p_c = 27.6\sqrt{D\gamma}/a^2$, where "$a$" is the plate width. If the applied pressure is less than this critical pressure, debonding will not occur. If it is greater than the critical pressure, debonding will commence at the midpoints of the plate boundary. To the best of our knowledge, experiments which measure the onset of debonding do not exist.

The implied assumption in solving both this problem and the example in Section 5.1 is that the region bonded to the substrate does not deform. Although the validity of this assumption may appear to be obvious, it has recently been shown (Ryan et al., 2008), in the context of a one-dimensional problem, that a one-sided bond to a rigid surface can cause unexpected deformation due to a combination of in-plane axial deformation and transverse shear deformation. In, for example, the case of a bridge structure this effect was shown to be important for a length-to-thickness ratio as large as about twenty. This behavior is in contrast to shear deformation which is only significant for lengths less than about 10 times the thickness. Although a corresponding theory of one-sided bonding for general two-dimensional plate problems is not available, we assume that a similar trend holds. Thus our analysis of the blister test can be expected to be valid for $a/H > 20$.

The particular advantage of using this method to solve this type of problem is that any finite element code can be used to solve a plate problem with known boundaries. The maximum value of the right-hand-side of Eq. (6) along the boundary can then be calculated. If this quantity is greater than γ then debonding will start; otherwise it will not.

6. Conclusions

It has been shown that a simple boundary condition can be used in place of energy minimization in order to solve adhesion problems with elastic plates with arbitrary contact geometries. This boundary condition represents a balance between mechanical energies with the work of adhesion. This formulation can be applied either to problems where the contact area is *a priori* unknown, or to problems in which the bonded area is predefined and it is the onset of debonding that is of interest. The simplified boundary condition is shown to be particularly useful and simple to use in the latter class of problems. Finite element programs can be used to solve for stresses in a standard manner. If the right-hand-side of Eq. (6) is greater/less than the work of adhesion, then debonding will/will-not occur.

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