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Remarks on formulating an adhesion problem using Euler's elastica (draft)

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Abstract

Three formulations for the problem of an elastica adhering to a rigid surface are discussed and compared. These include stationary principles, the surface integral of Eshelby's energy-momentum tensor, and the material (configurational) force balance. The configuration at static equilibrium is predicted in closed form for a pair of structures that arise in nano- and microscale applications.

Keywords: Elastic rod theory; Adhesion; Fracture mechanics; Variational calculus; Material force balance

1. Introduction

Rod theories have been increasingly used to study biological and synthetic nano- and microscale structures. Recent applications have ranged from synthetic gecko hair adhesives (Majidi et al., 2005) to stiction of high-aspect ratio polysilicon flexures in MEMS devices (de Boer and Michalske, 1999). In almost all such systems, the tendency for surfaces to adhere through electrostatic, capillary, or Van der Waals forces plays an important role in functionality and performance.

For the case of an elastic rod, adhesion will often result in a two-phase state in which the rod is either free or contacting an opposing surface (see Fig. 1(a)). Classical beam theories are not sufficient as they do not furnish the additional balance law needed to locate the phase interface. Here, I clarify two current solution techniques (Majidi et al., 2005; de Boer and Michalske, 1999; Plaut et al., 2001; Plaut et al., 2001; Glassmaker and Hui, 2004; Hui et al., 2002; Majidi et al., 2004; Glassmaker et al., 2004) and present a third method based upon material (configurational) forces (Kienzler, 1986; O'Reilly, in press).

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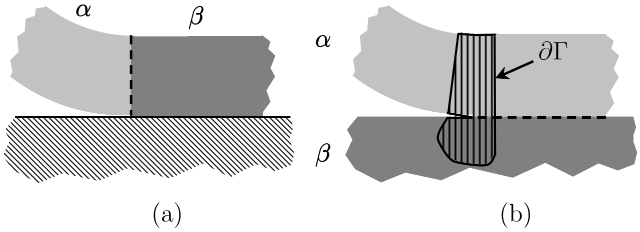

Fig. 1. (a) Composite representation of adhering elastica studied in Sections 3.1 and 3.3; (b) Single phase representation of elastica studied in Section 3.2; dashed line denotes phase interface.

2. Preliminaries

The current analysis concerns the adhesion of an elastic rod to a rigid half-space. The configuration of the rod is represented by a one-dimensional curve Ω parameterized by the coordinate ξ, which uniquely identifies material points on Ω and convects with the curve. Unlike classical beam theory, the rod is a two phase body composed of a non-contacting and contacting portion, denoted by Ω_α and Ω_β, respectively. Following the convention of Gurtin (1995), these two portions are assumed to be closed complementary subregions of Ω that share an interface $\mathscr{I} = \Omega_\alpha \cap \Omega_\beta$. Furthermore, it is assumed that Ω_α, Ω_β, and $\mathscr{I}$ are identified with the coordinates $[0, \gamma]$, $[\gamma, L]$, and $\{\gamma\}$, respectively, where L is the rod length and γ is generally unknown.

The position of a material point on Ω is defined by the vector $\mathbf{r} = \mathbf{r}(\xi)$, which maps $\xi \in [0, L]$ to the fixed Cartesian frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. Treating the rod as an elastica, $\mathbf{r}' = \cos\theta \mathbf{E}_1 + \sin\theta\ \mathbf{E}_2$, where the slope θ is defined as the angular deflection of the material curve from the $\mathbf{E}_1$ axis and the prime denotes the partial derivative w.r.t. ξ.

Here, we study functions of the form $\chi : \Omega \to \mathbb{R}$, some of them having a discontinuity at $\xi = \gamma$. It is convenient to decompose χ into continuous functions χ_α and χ_β on the domains Ω_α and Ω_β, respectively. In general, $\chi_\alpha = \chi_\alpha(\xi,\ \theta, \theta')$ and since $\theta(\xi)$ is prescribed on Ω_β, $\chi_\beta = \chi_\beta(\xi)$. At the phase interface $\xi = \gamma$, these functions are defined explicitly as their limit points. Lastly, the jump in χ is denoted by $[\![\chi]\!] = \chi_\beta(\gamma) - \chi_\alpha(\gamma)$.

The rod is subject to kinematic boundary conditions $\theta(0) = \theta_0$ and $\theta(\xi) = \theta_\beta(\xi)\ \forall\ \xi \in [\gamma, L]$, where the constant θ_0 and function $\theta_\beta(\xi)$ are prescribed. Additionally, there may be an isoperimetric constraint of the form

$$\int_0^L h(\theta)\,\mathrm{d}\xi = c, \tag{1}$$

where $h(\theta)$ and constant c are also prescribed (see, for example, the system illustrated in Fig. 3(a) where $h = \sin\theta$ and $c = \Delta$). For an elastica, the balance laws pertaining to the internal force $\mathbf{n}$ and bending moment $\mathbf{m}$ are $\mathbf{n}' = \mathbf{0}$ and $\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = \mathbf{0}$ and the constitutive law is $\mathbf{m} = EI\theta'\ \mathbf{E}_3$, where EI is the rod stiffness. A free body diagram of a rod element is provided in Fig. 2.

For the applications of interest, an external point force $\mathbf{F}$ may act at the base of the rod ($\xi = 0$). As $\mathbf{n}' = \mathbf{0}$, this implies that $\mathbf{n} = \mathbf{F}$ on Ω_α. Noting that $\mathbf{r}(L)$ is fixed, the potential energy associated with the external loads is

$$\mathbf{F} \cdot (\mathbf{r}(L) - \mathbf{r}(0)) = \int_0^L \mathbf{F} \cdot \mathbf{r}'\,\mathrm{d}\xi. \tag{2}$$

Other contributions to the potential energy of the system include the elastic strain energy per unit length of ξ

$$\psi(\theta') = \frac{1}{2} EI(\theta')^2, \tag{3}$$

and the work λh necessary to maintain the isoperimetric constraint, where the constant λ is an undetermined (Lagrange) multiplier. Although the work λh is associated with a conservative load, it is subsumed into the potential energy functional by the method of Lagrange multipliers (Lanczos, 1970). Lastly, the rod is subject to an external potential energy $u = u(\xi)$ associated with adhesion. Here, $u = 0$ on Ω_α and $u = -W_{\mathrm{ad}}$ on Ω_β, where W_{ad} is the energy of adhesion.

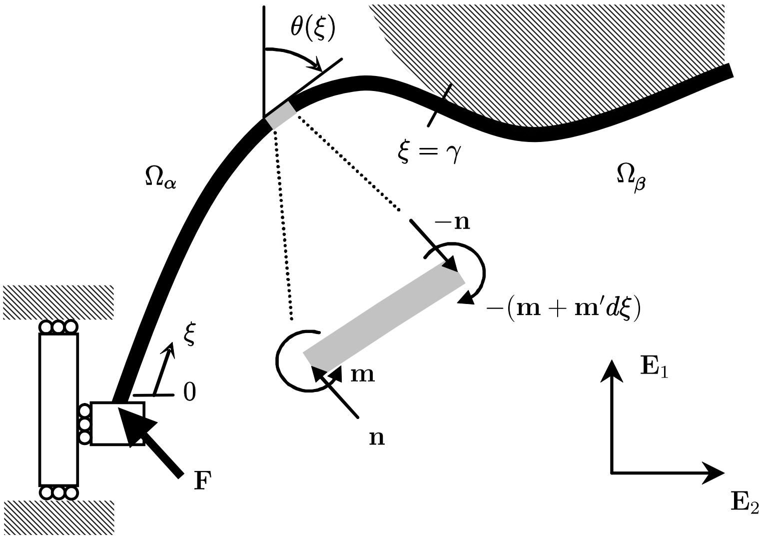

Fig. 2. Illustration of rod and free body diagram of a rod element.

In general, both u and ψ will have a discontinuity at the phase interface. Hence, the Lagrangian density for a static rod,

$$f(\xi, \theta(\xi), \theta'(\xi)) = \mathbf{F} \cdot \mathbf{r}' + \psi + \lambda h + u, \tag{4}$$

will be a piecewise continuous function with a jump at $\xi = \gamma$.

3. Natural boundary condition

Classical elastica theory yields a balance law for determining $\theta_\alpha(\xi)$ at static equilibrium for a given γ. Since such a solution is unique, the problem reduces to a one-dimensional search for the value of γ at equilibrium, denoted by γ^*. This equilibrium value is obtained from a natural boundary condition at the phase interface that can be derived from any one of the following three principles.

3.1. Principle of stationary potential (Kendall's theory of adhesion)

The total potential energy of the system is computed as

$$\Phi(\theta, \gamma) = \int_0^\gamma f_\alpha(\xi, \theta, \theta')\,\mathrm{d}\xi + \int_\gamma^L f_\beta(\xi)\,\mathrm{d}\xi. \tag{5}$$

According to the principle of stationary potential energy, the rod assumes a configuration $\{\theta^*(\xi), \gamma^*\}$ such that the action Φ is extremized.

Since θ_β is given, a necessary condition for Φ to be extremized is the Euler–Lagrange differential equation on Ω_α:

$$\frac{\partial f_\alpha}{\partial \theta_\alpha} - \frac{\partial}{\partial \xi}\left(\frac{\partial f_\alpha}{\partial \theta'_\alpha}\right) = 0. \tag{6}$$

Substituting the Lagrangian density yields the angular momentum balance for an elastica at static equilibrium. Eq. (6) follows from the stationarity of Φ with respect to variations of the form $\theta_\alpha \to \tilde{\theta}_\alpha = \theta_\alpha + \epsilon v$, where ϵ is infinitesimally small and the function $v = v(\xi)$ satisfies the boundary conditions $v(0) = v(\gamma) = 0$.

A natural boundary condition at $\xi = \gamma$ is obtained by admitting the additional variation $\gamma \to \tilde{\gamma} = \gamma + \epsilon s$, where s is a constant. Following a derivation similar to that used to solve a variable end point problem in the calculus of variations (Troutman, 1996), it can be shown that, at $\xi = \gamma$,

$$[\![f]\!] + \theta'_\alpha(\gamma)\left(\frac{\partial f_\alpha}{\partial \theta'_\alpha}\right)_\gamma = 0. \tag{7}$$

It is interesting to note that despite the discontinuity of f at $\xi = \gamma$, this result is equivalent to the second Weierstrass–Erdmann corner condition. The first Weierstrass–Erdmann corner condition, however, does not apply since $\theta(\gamma)$ is prescribed.

A more direct method for deriving the natural boundary condition (7) is to apply the Kendall theory of adhesion, which states that at equilibrium, $\mathrm{d}\Phi/\mathrm{d}\gamma = 0$ (Kendall, 1971; Plaut et al., 2001a; Plaut et al., 2001b). By the Leibniz rule,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\gamma} = -[\![f]\!] + \int_0^\gamma \left\{\frac{\partial f_\alpha}{\partial \theta}\frac{\partial \theta}{\partial \gamma} + \frac{\partial f_\alpha}{\partial \theta'}\frac{\partial \theta'}{\partial \gamma}\right\}\mathrm{d}\xi. \tag{8}$$

For a stationary function θ, the integrand is the derivative of $(\partial f_\alpha/\partial\theta')\,(\partial\theta/\partial\gamma)$. Also noting that $(\partial\theta/\partial\gamma)_0 = 0$ and $(\partial\theta/\partial\gamma)_\gamma = -\theta'(\gamma)$, it follows that the condition $\mathrm{d}\Phi/\mathrm{d}\gamma = 0$ reduces to (7) at $\xi = \gamma$.

Substituting the expression (4) for the Lagrangian density into (6), the Euler–Lagrange differential equation reduces to

$$EI\theta''_\alpha + \mathbf{F}\cdot(\sin\theta_\alpha\mathbf{E}_1 - \cos\theta_\alpha\mathbf{E}_2) - \lambda\frac{\partial h}{\partial\theta_\alpha} = 0. \tag{9}$$

Assuming that the substrate is flat (i.e., $\theta'_\beta(\xi) = 0$), the natural boundary condition (7) at $\xi = \gamma$ becomes

$$\frac{1}{2}EI(\theta'_\alpha(\gamma))^2 = W_{\mathrm{ad}}. \tag{10}$$

In summary, the equilibrium configuration $\{\theta^*(\xi), \gamma^*\}$ is the solution of the Euler–Lagrange differential Eq. (9), the prescribed boundary conditions, the isoperimetric constraint (1), and the natural boundary condition (10).

3.2. J-integral method

An alternative method for obtaining (10) is to treat the rod and opposing surface as two distinct phases and the interior edge of the corresponding phase interface as a crack tip (see Fig. 1(b)). Let $\partial\Gamma$ be any bounding surface containing the crack tip and define

$$\mathbf{J} = \int_{\partial\Gamma}\{W\mathbf{n} - (\nabla\mathbf{v})^{\mathrm{T}}(\mathbf{T}\mathbf{n})\}\,\mathrm{d}S, \tag{11}$$

to be the surface integral of Eshelby's energy momentum tensor. See Fig. 1(b) for a two-dimensional representation of the surface $\partial\Gamma$. Here, $\nabla\mathbf{v}$ is the three dimensional gradient of the displacement vector $\mathbf{v}$, $W = \mathrm{tr}(\mathbf{T}\boldsymbol{\epsilon})/2$ is the strain energy density, $\mathbf{T}$ is the Cauchy stress tensor, $\boldsymbol{\epsilon}$ is the infinitesimal strain tensor, and $\mathbf{n}$ is the unit normal to $\partial\Gamma$. It is well known in elastic fracture mechanics that $\mathbf{J}$ is subject to the conservation law

$$\mathbf{J}\cdot\mathbf{r}' = W_{\mathrm{ad}}, \tag{12}$$

where now W_{ad} is the work of adhesion per unit area (Eshelby, 1970). In studying the self-adhesion of a graphite sheet, (Glassmaker and Hui, 2004) show that for an elastica, (12) leads to the natural boundary condition (10).

3.3. Material force balance

The conservation law (12) is believed to be the outcome of a more fundamental balance involving material (configurational) forces (Gurtin, 1995; Maugin, 1995). Applying a material force balance to the composite Ω leads directly to a one-dimensional jump condition by which γ^* can be determined.

The material force of a static (non-moving) elastica is (Kienzler, 1986; O'Reilly, in press)

$$\mathsf{C} = \psi - \mathbf{m}\theta'\cdot\mathbf{E}_3 - \mathbf{n}\cdot\mathbf{r}', \tag{13}$$

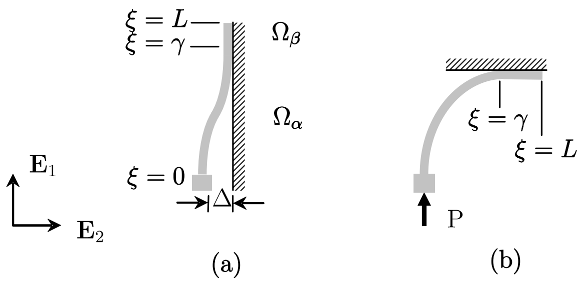

Fig. 3. Loading conditions for an adhering elastica in micro- and nanoscale systems: (a) stiction of a MEMS cantilever, (b) adhesion of a synthetic gecko hair.

where $\mathbf{m}$ is the bending moment and $\mathbf{n}$ is the internal force. The static material force balance law will be satisfied if the jump condition $[\![\mathsf{C}]\!] + \mathsf{B} = 0$ holds. Here, B is prescribed as a singular supply of material momentum (O'Reilly, in press). Physically, the parameter B represents the resistance of the interface to deformation (in this case translation of ξ) under the material forces. Since the external potential u is not included in the material force C, the resistance to interface translation corresponds to a jump $[\![u]\!]$.[1]

Using the definition of C in (13) and substituting the constitutive law $\mathbf{m} = EI\theta'\mathbf{E}_3$ and $\mathsf{B} = [\![u]\!]$ into the jump condition, we find

$$\left[\!\!\left[\psi + u - \theta' \frac{\partial \psi}{\partial \theta'}\right]\!\!\right] + \mathbf{F}_\gamma \cdot \mathbf{r}' = 0, \tag{14}$$

where $\mathbf{F}_\gamma = -[\![\mathbf{n}]\!]$ is the reaction force at $\xi = \gamma$. Next, substituting in the expressions for ψ and u and assuming a flat substrate $(\theta'_\beta(\xi) = 0)$,

$$\frac{1}{2} EI(\theta'_\alpha(\gamma))^2 + \mathbf{F}_\gamma \cdot \mathbf{r}' = W_{\mathrm{ad}}. \tag{15}$$

Applying the natural boundary condition (10), the material force balance implies the new result $\mathbf{F}_\gamma \cdot \mathbf{r}' = 0$. Physically, this suggests that the tangential component of the reaction force transmitted by the substrate cannot be singular at $\xi = \gamma$. Instead, it must be distributed over Ω_β. Such a result might be important to consider for applications involving friction and shear resistance.

4. Applications

Consider the system illustrated in Fig. 3(a), in which an adhering cantilever beam is subject to an isoperimetric constraint

$$\int_0^L \sin\theta \, \mathrm{d}\xi = \Delta. \tag{16}$$

The stationary function $\theta(\xi)$ is obtained by solving (9) and (16) for $h(\theta) = \sin\theta$, and $\theta_\alpha(0) = \theta_\alpha(\gamma) = \theta_\beta(\xi) = 0$. Assuming small angle deflection ($\sin\theta \approx \theta, \cos\theta \approx 1$), the equilibrium configuration $\{\theta^*(\xi), \gamma^*\}$ can be obtained with little effort by directly solving $\mathrm{d}\Phi/\mathrm{d}\gamma = 0$ (de Boer and Michalske, 1999; Hui et al., 2002; Majidi et al., 2004; Glassmaker et al., 2004). The final results are

$$\theta^*_\alpha(\xi) = \frac{6\Delta}{(\gamma^*)^2}\xi\left(1 - \frac{\xi}{\gamma^*}\right) \quad \text{and} \quad \gamma^* = L - \left(\frac{18EI\Delta^2}{W_{\mathrm{ad}}}\right)^{1/4}. \tag{17}$$

[1] In diffusion science, $\mathsf{B} = v/M$ where v is the velocity of the phase interface and the constant M is interface mobility [Shewmon, 1983]. Assuming that the energy stored in the interface $\mathscr{I}$ is invariant to translation, the mobility is infinite and so $\mathsf{B} = 0$. This convention, however, requires that the term ψ in (13) be replaced by the free energy density $\psi + u$.

When Δ is sufficiently large, γ^* vanishes. This implies that adhesion can be avoided if the elastic rod is spaced sufficiently far from the substrate.

A situation in which small angle deflection cannot be assumed is illustrated in Fig. 3(b). Here, the elastic rod is subject to an external load $\mathbf{F} = P\mathbf{E}_1$ at $\xi = 0$ and boundary conditions $\theta_\alpha(0) = 0$ and $\theta_\alpha(\gamma) = \theta_\beta(\xi) = \pi/2$. Solving (9) and (10), we find that

$$\theta_\alpha^*(\xi) = 2\mathrm{am}\left(\xi\sqrt{\frac{P + W_{\mathrm{ad}}}{2EI}}, \sqrt{\frac{2}{1 + W_{\mathrm{ad}}/P}}\right), \tag{18}$$

and

$$\gamma^* = \mathrm{F}\left(\frac{\pi}{4}, \sqrt{\frac{2}{1 + W_{\mathrm{ad}}/P}}\right)\sqrt{\frac{2EI}{P + W_{\mathrm{ad}}}}, \tag{19}$$

where $\mathrm{am}(u, k)$ is the Jacobi amplitude, $F(u, k)$ is the Jacobi integral of the first kind, and k is the modulus. Note that γ^* may be positive even when P is negative. The largest value of $-P$ for which γ^* is finite may be regarded as the peel strength of the elastic rod under this loading condition.

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