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On the stability of a rod adhering to a rigid surface: Shear-induced stable adhesion and the instability of peeling

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ABSTRACT

Using variational methods, we establish conditions for the nonlinear stability of adhesive states between an elastica and a rigid halfspace. The treatment produces coupled criteria for adhesion and buckling instabilities by exploiting classical techniques from Legendre and Jacobi. Three examples that arise in a broad range of engineered systems, from microelectronics to biologically inspired fiber array adhesion, are used to illuminate the stability criteria. The first example illustrates buckling instabilities in adhered rods, while the second shows the instability of a peeling process and the third illustrates the stability of a shear-induced adhesion. The latter examples can also be used to explain how microfiber array adhesives can be activated by shearing and deactivated by peeling. The nonlinear stability criteria developed in this paper are also compared to other treatments.

1. Introduction

The emergence of soft, miniaturized, and biologically inspired systems has led to particular interest in the dry adhesion of elastic rods. Relevant studies have examined the role of rod adhesion in microelectronic switches (Adams and McGreur, 2010), MEMS stiction (de Boer and Michalske, 1999), soft lithography stamp printing (Hui et al., 2002), muscle crossbridges (Stewart et al., 1987), nanotubes (Glassmaker and Hui, 2004), and gecko-inspired microfiber array adhesives (Majidi, 2009). These works are also related to extensive body of work on peeling problems (see Burridge and Keller, 1978; Plaut et al., 2001a,b; Podio-Guidugli, 2005, and references therein).

In the works on dry adhesion of rods, the conditions for static equilibrium are commonly derived from the stationarity of an energy functional that is composed of elastic bending energy and the work of adhesion. The purpose of the present paper is to develop stability and instability criteria for these equilibrium configurations. Of particular interest are the stability properties of shear-induced adhesion and a peeling process. These two examples are relevant to microfiber adhesive arrays. The stability of one of these examples can be exploited to promote adhesion while the instability can be used to deactivate the adhesion.

A prototypical problem featured in rod theory-based models for systems with adhesion is shown in Fig. 1. A rod of length ℓ is subject to a terminal load $\mathbf{P}$ and has a portion of length γ bonded to a surface. The equilibrium conditions for the rod dictate that the length γ depends on the components of $\mathbf{P}$, the geometric and material properties of the rod, and the

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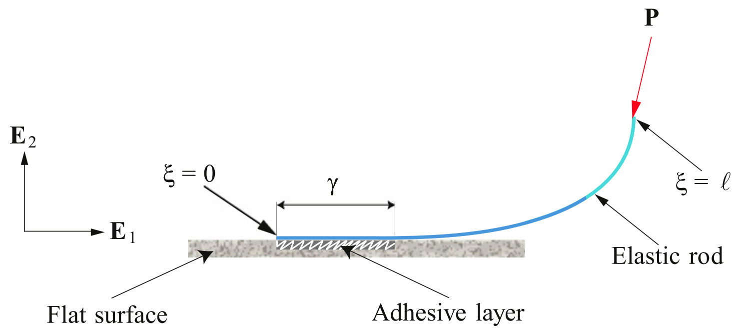

Fig. 1. Schematic of a rod of length ℓ a portion of which is glued to a horizontal surface.

adhesion energy ω. Dating to the seminal work by Kendall (1971), variational principles can be used to show that the equilibrium configuration of the rod is such that the potential energy Π is extremized with respect to changes in $\gamma : d\Pi/d\gamma = 0$. Some authors, e.g., de Boer and Michalske (1999) and Mastrangelo and Hsu (1993), also examine the second derivative $d^2\Pi/d\gamma^2$ and postulate minimization of Π as a function of γ as a stability criterion.

We concentrate our attention on Euler's theory of the elastica as this is the predominant rod theory featured in the literature on dry adhesion of rods. The criteria are developed by exploiting classical investigations of the second variation by Legendre and Jacobi.[1] Our work is intimately related to the extension of these classical works to develop stability criteria for tree-like structures composed of elastic rods which was recently presented in O'Reilly and Peters (2012). A further consequence of our treatment is the ability to relate the criteria to nonlinear treatments of buckling instabilities in rods (which can be found in works by Born, 1906; Jin and Bao, 2008; Maddocks, 1984; Manning, 2009; O'Reilly and Peters, 2011, 2012 and many others).

An outline of this paper is as follows: In Section 2, the relevant background material on adhesion and the elastic rod theory are introduced, and in Section 3 it is used to formulate a variational principle for the problem of an elastica which is in contact with a rigid surface. Particular attention is also placed on compatibility conditions which must be satisfied at the edge of the contact region. Section 4 is devoted to discussions on stability criteria. In particular, two stability criteria are developed in Sections 4.3 and 4.4, respectively. The first criterion is a necessary condition for stability, which we denote by N1. As in traditional treatments of buckling instabilities, the criterion N1 features the search for a bounded solution to a Riccati equation but with two subtle, yet important, differences. First, the domain of integration for the Riccati equation depends on the loading, adhesive strength, and geometric and material properties of the rod. Second, the solution to the Riccati equation must also satisfy a terminal inequality in order for N1 to hold. Our work in this respect can be viewed as an extension of the aforementioned nonlinear treatments of buckling instabilities to cases where adhesion is present. Section 4.5 of the paper is devoted to a discussion of a stability criterion where Π as a function of γ is examined.

For some of the problems of interest in this paper, we were fortunate to be able to establish a sufficient condition for stability, which we label S1, by slightly modifying a result of Gelfand and Fomin (1963). The criteria N1 and S1 are illustrated in Section 5 by applying them to three examples that have received significant attention in the literature in the past three decades. We also illustrate how the criteria N1 and S1 compare to stability treatments where the potential energy is expressed as a function of the adhesion length and the system parameters. The paper closes with a discussion of possible extensions to the criteria and open issues. The paper contains three appendices which are devoted to a proof of the criterion N1 and the establishment of explicit solutions for the deformed shape of the rod using elliptic functions.[2]

2. Background on adhesion energy and the elastica

The elastica is the simplest nonlinear theory of a deformable rod. Here we are interested in using this theory to model the adhesion of a rod-like body to a surface. We assemble all the background in this section that is required to formulate variational principles for the problems of interest and to examine the stability of the resulting equilibrium configurations.

2.1. Kinematics

When using Euler's theory of the elastica, the centerline $\mathcal{C}$ of the rod is modeled as an inextensible material curve which is free to move on a plane. This curve is assumed to be subject to external forces and moments and have a resistance to bending proportional to its curvature. As shown in Fig. 2, the material points of $\mathcal{C}$ are identified using a convected coordinate ξ and the position vector of a material point in the present configuration of the elastica is defined by

[1] The relevant background on the variational calculus used in the present paper can be found in the texts of Bolza (1904), Ewing (1969), and Gelfand and Fomin (1963).

[2] Relevant background on the use of elliptic functions to solve the deformed shapes of the elastica subject to various terminal loadings can be found in Frisch-Fay (1962) and Love (1944).

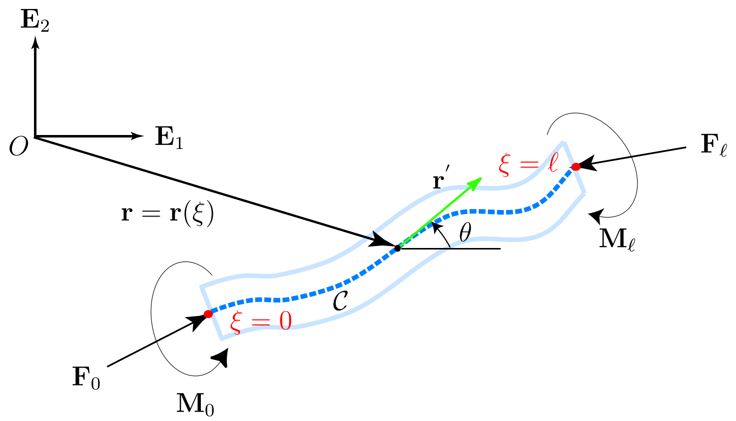

Fig. 2. Schematic of an elastica which is subject to a terminal force $\mathbf{F}_0$ and terminal moment $\mathbf{M}_0$ at $\xi=0$ and a terminal force $\mathbf{F}_\ell$ and terminal moment $\mathbf{M}_\ell$ at the end $\xi=\ell$.

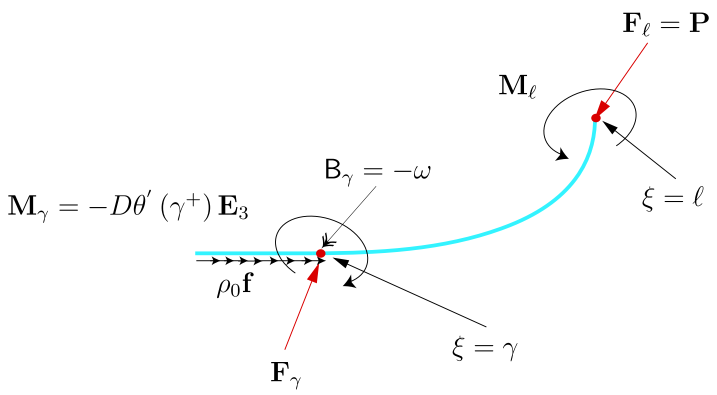

Fig. 3. Schematic of the singular supplies of linear momentum $\mathbf{F}$, angular momentum $\mathbf{M}$ and material force B acting on the rod shown in Fig. 1. A subscript on these quantities is used to denote the material point at which they are applied: e.g., B_γ and $\mathbf{M}_\ell$.

the vector-valued function $\mathbf{r}=\mathbf{r}(\xi)$. The mass density per unit length of $\mathcal{C}$ is denoted ρ_0. A fixed Cartesian basis $\{\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3\}$ and a fixed origin O are defined so that $\mathbf{r}=x\mathbf{E}_1+y\mathbf{E}_2$.

The unit tangent vector $\partial\mathbf{r}/\partial\xi$ to $\mathcal{C}$ can be used to define an angle θ

$$\mathbf{r}' = \cos(\theta)\mathbf{E}_1+\sin(\theta)\mathbf{E}_2, \tag{1}$$

where the prime is used to denote the partial derivative with respect to ξ. The unsigned curvature κ of $\mathcal{C}$ is identified with θ'. Both $\mathbf{r}$ and θ are assumed to be continuous functions of ξ.

For the problems of interest in this paper we will expect to find points $\xi=\zeta$ where the fields associated with the elastica experience discontinuities. These discontinuities can be induced by the application of point (or concentrated) loads $\mathbf{F}_\zeta$ and point moments $\mathbf{M}_\zeta$ relative to a point on the centerline, discontinuities in the slope θ', or the presence of a material force B_ζ induced by an adhesion energy (see Fig. 3). To help formulate such problems, we now define the following limits for any function $\chi=\chi(\xi,\theta,\theta')$:

$$\chi(\zeta^-) = \lim_{\xi\nearrow\zeta}\chi(\xi,\theta(\xi),\theta'(\xi)), \quad \chi(\zeta^+) = \lim_{\xi\searrow\zeta}\chi(\xi,\theta(\xi),\theta'(\xi)). \tag{2}$$

The jump in the function χ can be represented as

$$[\![\chi]\!]_\zeta = \chi(\zeta^+)-\chi(\zeta^-). \tag{3}$$

As a consequence of the continuity of $\mathbf{r}$ and θ, $[\![\mathbf{r}']\!]_\xi=\mathbf{0}$.

2.2. Strain energy, kinetics, and adhesion energy

Associated with the elastica are the assigned force per unit length $\rho_0\mathbf{f}$ and assigned moment per unit length $\mathbf{m}_a$. In addition, one has the contact force $\mathbf{n}=n_1\mathbf{E}_1+n_2\mathbf{E}_2$ and bending moment $\mathbf{m}=m\mathbf{E}_3$. The moment and the strain energy function $\rho_0\phi$ per unit length of $\mathcal{C}$ have the celebrated constitutive equations

$$\mathbf{m} = D\theta'\mathbf{E}_3, \quad \rho_0\phi = \frac{D}{2}(\theta')^2, \tag{4}$$

where D is the flexural rigidity. We supplement the fields $\mathbf{n}$ and $\mathbf{m}$ with the material contact force C

$$\mathsf{C} = \rho_0\phi - \mathbf{n}\cdot\mathbf{r}' - \mathbf{m}\cdot\theta'\mathbf{E}_3, \tag{5}$$

and the assigned material force b.

We will consider problems where a segment of the rod will be in contact with a substrate. To model these cases, we define an energy of adhesion ω per unit length of contact between a straight rod and a flat rigid substrate. Physically, ω is the external mechanical work required to separate the rod from equilibrium contact to infinite separation from the substrate. During the separation process, external work is aided by elastic restoring forces and resisted by the interfacial forces. We follow classical treatments (see Israelachvili, 1992; Kendall, 1971) in assuming that ω vanishes when the rod is separated from the substrate, even when the separation is infinitesimally small. For the cases of interest in this paper, the rod has a flat, ribbon-like cross-section and ω can be simply defined as the product of the cross-sectional width and the work of adhesion.

2.3. Balance laws and jump conditions

The local form of the static balance laws for the elastica are the balance of material momentum, balance of linear momentum, and balance of moment of momentum

$$\mathsf{C}' + \mathsf{b} = 0, \quad \mathbf{n}' + \rho_0\mathbf{f} = \mathbf{0}, \quad \mathbf{m}' + \mathbf{m}_a + \mathbf{r}' \times \mathbf{n} = \mathbf{0}. \tag{6}$$

As discussed in O'Reilly (2007), the force b is prescribed such that the local balance of material momentum $(6)_1$ is identically satisfied.

At a point $\xi = \zeta$ where a kinematic discontinuity is present or a singularity in the applied fields $\mathbf{M}_\zeta$, $\mathbf{F}_\zeta$, or B_ζ occurs, we have the following jump conditions:

$$[\![\, \mathbf{n} \,]\!]_\zeta + \mathbf{F}_\zeta = \mathbf{0}, \quad [\![\, \mathsf{C} \,]\!]_\zeta + \mathsf{B}_\zeta = 0, \quad [\![\, \mathbf{m} \,]\!]_\zeta + \mathbf{M}_\zeta = \mathbf{0}. \tag{7}$$

The conditions are also helpful in establishing boundary conditions. For example, for the situation shown in Fig. 2, we can use $(7)_{1,3}$ to infer that $\mathbf{n}(0^+) = -\mathbf{F}_0$ and $\mathbf{m}(\ell^-) = \mathbf{M}_\ell$. In addition, $(7)_2$ can be shown to be equivalent to an adhesion boundary condition with $\mathsf{B}_\gamma = -\omega$.[3]

3. Formulation of the problem of an elastica in contact with a rigid surface

In the applications considered in this paper we assume that the assigned force $\rho_0\mathbf{f} = \mathbf{0}$ and the assigned moment $\mathbf{m}_a = \mathbf{0}$. It follows that the local balance laws (6) simplify dramatically to the statements that the force $\mathbf{n}$ and moment $\mathbf{m} + \mathbf{r} \times \mathbf{n}$ are piecewise constant throughout segments of the elastica.

For a rod which is in contact with a rigid substrate, the total potential energy of the rod will be composed of the potential energy of terminal forces and moments, the integral of the strain energy per unit length, and the adhesion energy per unit length $-\omega$. It is convenient to decompose the potential energy into the sum of the elastic potential energy in the non-contacting and contacting sections. Modulo an additive constant, the resulting expression for the total potential energy Π is

$$\Pi = \int_\gamma^\ell \left\{ \frac{D}{2}(\theta')^2 - \mathbf{n}\cdot\mathbf{r}' \right\} d\xi + \int_0^\gamma \{ -\mathbf{n}\cdot\mathbf{E}_1 - \omega \} d\xi. \tag{8}$$

In writing (8), we noted that in the contact regions, $\mathbf{r}' = \mathbf{E}_1$ and $\theta = 0$.

It is evident that the adhesion energy ω is subtracted from the last term in (8). This is because ω is defined as the work of the adhesive and elastic restoring forces during interfacial detachment. Alternatively, the adhesion may be represented as a surface potential by also adding $\omega\ell$ to Π. This is accomplished by eliminating ω in the second integrand and adding it to the first integrand.

3.1. Variations and compatibility conditions

In the sequel, the behavior of the functional (8) with respect to variations in θ and γ will be computed

$$\theta = \theta(\xi,\epsilon) = \theta^*(\xi) + \epsilon\eta(\xi), \quad \gamma = \gamma(\epsilon) = \gamma^* + \epsilon\mu. \tag{9}$$

That is, the respective variations in θ and γ are

$$\delta\theta = \epsilon\eta, \quad \delta\gamma = \epsilon\mu. \tag{10}$$

[3] For further details on this boundary condition and its relation to material momentum, we refer the reader to Majidi (2007) and O'Reilly (2007). In O'Reilly's paper, $\omega = W_{\mathrm{ad}}$.

In order to avoid confusion where it may arise, an asterisk is often used to denote the equilibrium solution. In the contact region, $\theta^* = \theta = 0$, and so

$$\eta(\xi) = 0 \; \forall \xi \in [0,\gamma). \tag{11}$$

We need to establish a set of compatibility equations for the first and second derivatives of $\theta(\xi = \gamma^* + \epsilon\mu, \epsilon)$ with respect to ϵ evaluated at $\epsilon = 0$. The desired set of compatibility conditions are obtained by taking the first and second derivatives of $(9)_1$ with respect to ϵ and then setting $\epsilon \to 0$

$$[\![\mu\theta^{*'} + \eta]\!]_\gamma = 0, \quad [\![\mu^2\theta^{*''} + 2\mu\eta']\!]_\gamma = 0. \tag{12}$$

Compatibility conditions of the form (12) can be found in Majidi and Wan (2010), O'Reilly and Peters (2012), and Seifert (1991). They express the restrictions that variations in $\theta(\gamma^{\pm})$ and γ are not always independent.

3.2. Static balance laws

By considering variations of the form $(10)_1$ and keeping γ fixed, we find that the equation $d\Pi/d\epsilon|_{\epsilon=0} = 0$ leads to the balance equation

$$D\theta^{*''} = P_1, \tag{13}$$

and, in the event that $\theta^*(\ell)$ is not prescribed, the natural boundary condition

$$\theta^{*'}(\ell) = 0. \tag{14}$$

In the balance law (13),

$$P_1 = \mathbf{n} \cdot (\sin(\theta^*)\mathbf{E}_1 - \cos(\theta^*)\mathbf{E}_2). \tag{15}$$

As anticipated, these results are consistent with the developments in Section 2.3.

3.3. Adhesion boundary conditions

The natural boundary condition at the edge of the region of adhesive contact is obtained by applying the variations (10). After differentiating the expression for the functional Π with respect to ϵ, using the Leibniz rule, taking the limit $\epsilon \to 0$, using (13) and (14), and then setting the resulting expression to 0, we find that

$$\left(\left[\!\left[\frac{D}{2}(\theta^{*'})^2 - \mathbf{n}\cdot\mathbf{r}' \right]\!\right]_\gamma - \omega \right)\mu = 0. \tag{16}$$

The condition (16) must hold for all μ. Whence we find the adhesion boundary condition[4]

$$\left[\!\left[\frac{D}{2}(\theta^{*'})^2 - \mathbf{n}\cdot\mathbf{r}' \right]\!\right]_\gamma = \omega. \tag{17}$$

This boundary condition can be further simplified by noting that $[\![\mathbf{r}']\!]_\gamma = \mathbf{0}$. Thus, we can use $(7)_2$ to write

$$\left[\!\left[\frac{D}{2}(\theta^{*'})^2 \right]\!\right]_\gamma + \mathbf{F}_\gamma \cdot \mathbf{E}_1 = \omega. \tag{18}$$

In the absence of shear adhesion (i.e., $\mathbf{F}_\gamma$ is normal to the surface and so $\mathbf{F}_\gamma \cdot \mathbf{E}_1 = 0$) or when the shear traction is distributed along the interface, the boundary condition (18) is the same natural boundary condition previously derived in Seifert (1991) and Majidi (2007) and corresponds to the jump in material momentum $(7)_2$ with $\mathsf{B}_\gamma = -\omega$.

In addition to a force $\mathbf{F}_\gamma$ and material force B_γ at the edge of the region of adhesive contact, a moment $\mathbf{M}_\gamma$ can also be present. This moment is computed from $(7)_3$

$$\mathbf{M}_\gamma = D\theta^{*'}(\gamma^-)\mathbf{E}_3 - D\theta^{*'}(\gamma^+)\mathbf{E}_3. \tag{19}$$

An example of $\mathbf{M}_\gamma$ can be seen in Fig. 3. We also note that the moment $\mathbf{M}_\gamma$ is identical to the adhesion moment discussed in Pamp and Adams (2007).

3.4. Remarks

Clearly, the necessary conditions for the vanishing of the first variation of Π are the local balance laws and jump conditions discussed in Section 2.3. When specialized to the loading conditions presented in Fig. 1, these governing equations reduce to the balance law (13), natural boundary condition (14), and adhesion-controlled jump condition (18). The variational formulation naturally leads us to examine the second variation in order to establish necessary conditions

[4] If the rod were extensible, then $[\![\mathbf{n}\cdot\mathbf{r}']\!]_\gamma$ would be due to the jump in the stretch of the centerline across the discontinuity. For examples where this situation arises, see Kendall (1971) and Majidi and Adams (2010).

for stability of the elastica. In the following section, we employ the variations (9) and compatibility conditions (12) along with classical transformations by Legendre and Jacobi to simultaneously address adhesion and buckling instabilities.

4. Conditions for stability from the second variation

Necessary conditions for the functional Π to be minimized includes the vanishing of the first variation and the non-negativity of the second variation. We now explore these conditions and use them to establish a necessary condition, which we label N1, for stability. For some cases, we are able to establish a sufficient condition, which is referred to as S1, for stability. This section of the paper concludes with a discussion of an alternative criterion for stability which is based on computing Π as a function of γ. All three stability criteria will be illuminated by examples in Section 5.

4.1. A representation for the second variation

To establish an expression for the second variation of the potential energy Π, we again consider variations of the form (9) and evaluate $d^2\Pi/d\epsilon^2|_{\epsilon=0}$. After some rearranging we find that the second variation has a simple additive decomposition

$$\left.\frac{d^2\Pi}{d\epsilon^2}\right|_{\epsilon=0} = \int_\gamma^\ell (D\eta'\eta' + P\eta^2)\,d\xi - 2[\![D\theta^{*'}\eta' + P_1\eta]\!]_\gamma \mu - [\![D\theta^{*''}\theta^{*'} + P_1\theta^{*'}]\!]_\gamma \mu^2, \tag{20}$$

where

$$P = \mathbf{n}\cdot(\cos(\theta^*)\mathbf{E}_1 + \sin(\theta^*)\mathbf{E}_2). \tag{21}$$

To simplify (20), we can use (12)

$$0 = \mu\theta^{*'}(\gamma^+) + \eta(\gamma^+), \quad 0 = \mu\theta^{*''}(\gamma^+) + 2\eta'(\gamma^+), \tag{22}$$

where we have used the fact that, because θ is prescribed in the contact region, $\eta(\gamma^-) = 0$. In addition, we also appeal to the balance law (13) to eliminate $D\theta^{*''}$ from (20)

$$J = \left.\frac{d^2\Pi}{d\epsilon^2}\right|_{\epsilon=0} = \int_\gamma^\ell (D\eta'\eta' + P\eta^2)\,d\xi + P_1\theta^{*'}(\gamma^+)\mu^2. \tag{23}$$

We now follow Legendre and add the following term to (23)

$$\int_\gamma^\ell \frac{d}{d\xi}(\eta^2 w)\,d\xi - [\eta^2 w]_\gamma^\ell = 0. \tag{24}$$

Manipulating the resulting expression for J in a standard manner, we see that provided a solution $w(\xi)$ to the following Riccati equation can be found:

$$\frac{\partial w}{\partial \xi} + P - \frac{w^2}{D} = 0, \tag{25}$$

we can then express J as

$$J = \int_\gamma^L D\left\{\eta' + \left(\frac{w}{D}\right)\eta\right\}^2 d\xi + J_2, \tag{26}$$

where

$$J_2 = P_1\theta^{*'}(\gamma^+)\mu^2 - \eta^2(\ell)w(\ell) + \eta^2(\gamma^+)w(\gamma^+). \tag{27}$$

We will choose the initial condition $w(\ell) = 0$ and note that the existence of a bounded $w(\xi)$ on the interval $[\gamma,\ell]$ implies that the integral in (26) is non-negative. We now seek necessary conditions for $J \ge 0$.

4.2. Riccati and Jacobi equations

To continue, it is fruitful to employ a Jacobi transformation

$$w = -D\frac{u'}{u}. \tag{28}$$

As is well-known, this transformation produces a Jacobi differential equation for $u(\xi)$ from (25)

$$Du'' - Pu = 0. \tag{29}$$

We consider solutions $u(\xi)\forall\xi \in [\gamma,\ell]$ to (29) which satisfy the initial conditions

$$u(\ell) = 1, \quad u'(\ell) = 0. \tag{30}$$

Note that this initial condition is equivalent to $w(\ell)=0$. If the solution $u(\xi_c)=0$ for some ξ_c, then the point ξ_c is said to be conjugate to $\xi=\ell$.[5]

To relate unbounded solutions of the Riccati equation to conjugate points, we recall a theorem from Reid (1972): bounded solutions to the Riccati equation (25) for $w(\xi)$ on a given interval exist if, and only if, a solution $u(\xi)$ for the corresponding Jacobi differential equation (29) exists on the same interval with $u(\xi)\neq 0$ and w given by (28). Thus, at a conjugate point, the solution to the Riccati equation becomes unbounded: $\lim_{\xi\to\xi_c}|w(\xi)|=\infty$ (see Fig. 4). On the other hand, the existence of a bounded solution $w(\xi)\forall\xi\in[\gamma,\ell]$ is equivalent to the non-existence of conjugate points to $\xi=\ell$ in the interval $[\gamma,\ell]$ for the solution $u(\xi)$ to (29).

4.3. The condition N1

We have now compiled all the needed background to state the necessary condition for stability N1. Part of this criterion pertains to the buckling instability of the rod. The second part of the criterion is intimately related to the kinematics at the adhesion point $\xi=\gamma$. The proof of the criterion is discussed in Appendix A.

Condition N1. *If a solution $\{\theta^*(\xi),\gamma^*\}$ to the boundary-value problem minimizes Π then the solution $w(\xi)\forall\xi\in(\gamma^*,\ell]$ to the boundary-value problem*

$$\frac{\partial w}{\partial \xi}+P-\frac{w^2}{D}=0, \quad w(\ell)=0, \tag{31}$$

cannot become unbounded in the interval $[\gamma,\ell]$ and the following inequality must be satisfied:

$$\theta^{*'}(\gamma^+)(\theta^{*'}(\gamma^+)w(\gamma^+)+P_1)\geq 0. \tag{32}$$

The adhesion boundary condition (18) can often be used to express $\theta^{*'}(\gamma^+)$ in terms of ω. In this instance, the condition (32) can then be interpreted as a condition on the relative strength of the adhesive to the loading P_1. On the other hand, the existence of a solution to the Riccati equation (31) implies that the rod has not buckled. For completeness, we also note that the criterion N1 is the counterpart of the criteria L1 and B3 that were developed for tree-like structures composed of elastic rods in O'Reilly and Peters (2012).

4.4. The condition S1

For many applications of the stability criterion, $\theta^{*'}(\gamma^+)=0$. In this case, we can strengthen the condition N1 to yield a sufficient condition, which we denote by S1, for stability. To elaborate, when $\theta^{*'}(\gamma^+)=0$ the compatibility equation $(12)_1$ implies that $\eta(\gamma^+)=0$. Hence, the boundary term J_2 defined in (27) can be set to zero by choosing the initial condition

$$w(\ell)=0. \tag{33}$$

Clearly, J is not a function of μ for the case at hand. As a result, the forthcoming criterion only pertains to perturbations in θ: perturbations to γ are not considered.

Referring to Theorem 3 of Section 26 in Gelfand and Fomin (1963), we can now readily establish a sufficient condition for positive definiteness of J and, hence, a sufficient condition for stability.[6] In the interests of brevity, we merely state the criterion:

Condition S1. *Consider the case $\theta^{*'}(\gamma^+)=0$. If a solution $\{\theta^*(\xi),\gamma^*\}$ to the boundary-value problem is such that either*

(i) *a bounded solution $w(\xi)\forall\xi\in[\gamma^*,\ell]$ to (25) can be found where $w(\ell)=0$, or*
(ii) *there are no points conjugate to $\xi=\ell$ in the interval $[\gamma^*,\ell]$,*

then $\{\theta^*(\xi),\gamma^*\}$ is stable.

Clearly, the development of a stability criterion in this case is identical in all but one respect to the case of a rod fixed at one end and subject to a terminal load **P** at the other.[7] The distinction from this classical problem is that the length $\ell-\gamma$ of the beam is typically a (nonlinear) function of **P**, D, and ω.

[5] Our definition of the conjugate point differs from the traditional definition as the latter applies to the case where both endpoints are fixed.

[6] Gelfand and Fomin's proof pertains to the fixed-fixed case. It requires some minor modifications to deal with the fixed-free case of interest here and these modifications are outlined in Peters (2011).

[7] I.e., the problem of a terminally loaded fixed-free strut.

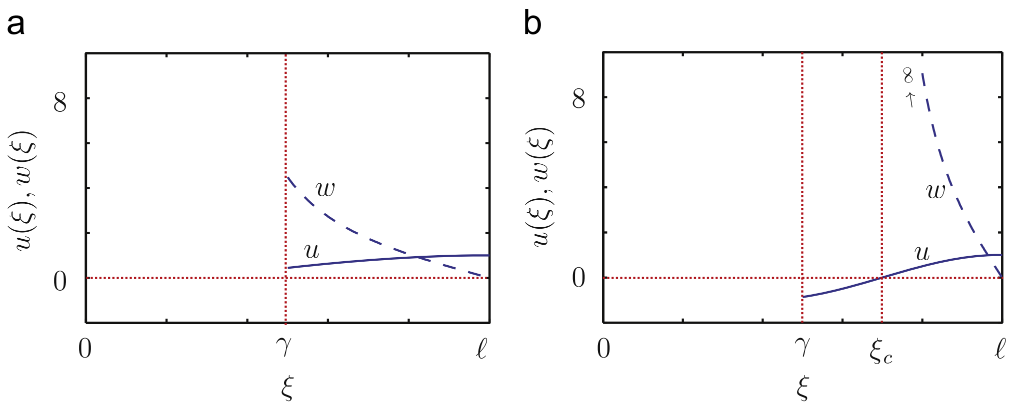

Fig. 4. Schematic of solutions $u(\xi)$ to a Jacobi equation (29) and the corresponding solutions $w(\xi)$ to a Riccati equation (25). In (a) there is no conjugate point to $\xi=\ell$ in the interval $[\gamma,\ell]$ and in (b) there is a conjugate point ξ_c to $\xi=\ell$ in the interval $[\gamma,\ell]$ where $w(\xi)\nearrow\infty$ as $\xi\searrow\xi_c$ and $w(\xi)\searrow-\infty$ as $\xi\nearrow\xi_c$.

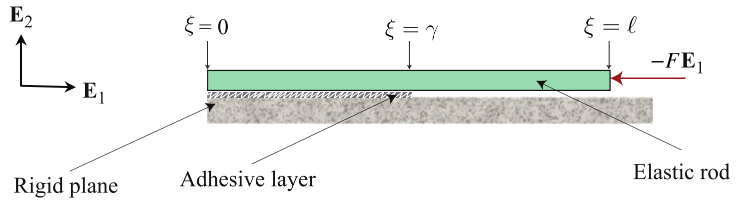

Fig. 5. An axially loaded elastic rod of length ℓ. A portion of the rod $\xi\in[0,\gamma)$ is stuck to the rigid plane.

4.5. *The potential energy as a function of γ*

An alternative stability criterion is often used in the literature (see, e.g., Mastrangelo and Hsu, 1993). Here, the solution for θ^* is substituted into the expression for Π in (8). The resulting expression for Π, which we denote by Π_c, depends on γ. Our examples indicate that solving $d\Pi_c/d\gamma=0$ for γ yields γ^*. The stability criterion is simply that Π_c has a minimum at $\gamma=\gamma^*$

$$\left.\frac{d^2\Pi_c}{d\gamma^2}\right|_{\gamma=\gamma^*}\begin{cases}>0 & \text{for stability,}\\ =0 & \text{inconclusive,}\\ <0 & \text{for instability.}\end{cases} \tag{34}$$

Our examples indicate, but have not been able to prove, that $d^2\Pi_c/d\gamma^2|_{\gamma=\gamma^*}>0$ is equivalent to the condition (32) which features in the condition N1. The main hinderance in such a proof is the compatibility condition $(22)_1$: $0=\mu\theta^{*'}(\gamma^+)+\eta(\gamma^+)$. This condition states that it is generally not possible to vary γ without varying $\theta(\gamma^+)$: a dependency that is easily visualized. Hence, it is not obvious that the derivative with respect to γ featuring in (34) should correspond to a variation $\delta\gamma$.

5. Examples

The stability criteria N1 and S1 and the criterion (34) featuring Π_c are illustrated using three loading conditions that commonly arise in engineered systems. All of these examples represent special cases of the general loading condition illustrated in Fig. 1. In the first example, $\theta^{*'}(\gamma^+)=0$ and so we can appeal to the necessary condition N1 to conclude instability and the sufficiency condition S1 to draw conclusions about stability. However, for the remaining examples, $\theta^{*'}(\gamma^+)>0$ and so we are restricted to examining only the necessary condition N1.

5.1. *A rod in friction–adhesion with a rigid flat surface*

Our first example is shown in Fig. 5. Here a rod of length ℓ has a section of length γ which is restrained by friction from moving on a horizontal surface. The contact between the rod and the surface is idealized as adhesionless: $\omega=0$ in this problem. An applied force $-F\mathbf{E}_1$ acts at the material point $\xi=\ell$. Clearly, if the applied force F is too large then the unattached section of the rod will buckle.

To analyze this problem, we first solve the balance laws with the help of the appropriate boundary conditions and find the trivial solution

$$\mathbf{n}(\xi)=-F\mathbf{E}_1,\quad \theta^*(\xi)=0. \tag{35}$$

This solution is statically admissible for all values of F and all adhesion lengths $\gamma > 0$. To examine the possibility of buckling, we establish the appropriate single Riccati and Jacobi equations, from (25) and (29),

$$\frac{\partial w}{\partial \xi} = F + \frac{w^2}{D}, \quad Du'' + Fu = 0, \quad \xi \in (\gamma, \ell]. \tag{36}$$

The boundary conditions for these differential equations are, respectively,

$$w(\ell) = 0, \quad u(\ell) = 1, \quad u'(\ell) = 0. \tag{37}$$

The solutions of the pair of boundary-value problems are

$$u(\xi) = \cos\left(\sqrt{\frac{F}{D}}(\ell - \xi)\right), \quad w(\xi) = \sqrt{FD}\tan\left(\sqrt{\frac{F}{D}}(\ell - \xi)\right). \tag{38}$$

The Riccati equation has a bounded solution and, equivalently, there are no conjugate points to $\xi = \ell$ in the interval $[\gamma, \ell)$ provided $F < F_{\text{crit}}$[8]:

$$F_{\text{crit}} = \frac{\pi^2 D}{4(\ell - \gamma)^2}. \tag{39}$$

We conclude with the help of S1 that the straight configuration $\theta^* = 0$ is stable provided $F < F_{\text{crit}}$ and unstable otherwise. Notice that instability occurs when the rod buckles and is independent of the adhesive.

Now suppose that $F > F_{\text{crit}}$. It follows that $u(\xi_c) = 0$ for $\xi = \xi_c$. That is, $\xi = \xi_c$ is the conjugate point to $\xi = \ell$. We observe from (38) that $w(\xi) \nearrow \infty$ as $\xi \searrow \xi_c$ as expected from our earlier discussion in Section 4.2. An example of this situation when $\gamma = 0.5\ell$ is shown in Fig. 4(b).

We next examine the criterion (34). For the problem at hand, we substitute the solutions (35) for θ and $\mathbf{n}$ in the expression (8) for Π to find that

$$\Pi_c = F\ell. \tag{40}$$

Clearly, the derivatives of Π_c with respect to γ are zero. Hence, the criterion (34) is not satisfied. It is also interesting to notice that the criterion featuring Π_c is insensitive to the buckling instabilities in this problem.

5.2. Shear peeling

Next, we consider a load $\mathbf{P} = P\mathbf{E}_2$ applied at the free end $\xi = \ell$ of the elastica. According to the adhesion boundary condition (18), the elastica peels when the curvature $\theta^{*'}(\gamma^+)$ induced by the bending moment exceeds the critical value $\sqrt{2\omega/D}$. This relates to the "moment-discontinuity method" introduced by Pamp and Adams (2007) to study the peeling of elastic plates and rods.

The balance laws and boundary conditions are easily inferred from (13), (14), and (18) over the domain $\xi \in (\gamma, \ell]$

$$\mathbf{n}(\xi) = P\mathbf{E}_2, \quad D\theta^{*''} = -P\cos(\theta^*),$$

$$\theta^*(\gamma^+) = 0, \quad \theta^{*'}(\ell) = 0, \quad \frac{D}{2}(\theta^{*'}(\gamma^+))^2 = \omega. \tag{41}$$

These equations are used to solve $\theta^*(\xi)$ and determine the contact length γ^* at static equilibrium. Following the derivations in Appendix B, we evaluate the boundary-value problem $(41)_{2-5}$ to obtain the following solutions for θ^* and γ^*:

$$\theta^* = 2\,\mathrm{am}(v|m) - \frac{\pi}{2},$$

$$\gamma^* = \ell - \left\{\frac{1}{\sqrt{m}}\mathrm{K}(\mu) - \mathrm{F}\left(\frac{\pi}{4}\middle|m\right)\right\}\sqrt{\frac{2D}{P+\omega}}. \tag{42}$$

In the first of these equations, $\mathrm{am}(v|m)$ is the Jacobi amplitude evaluated at

$$v = (\xi - \gamma^*)\sqrt{\frac{P+\omega}{2D}} + \mathrm{F}\left(\frac{\pi}{4}\middle|m\right), \tag{43}$$

$\mathrm{F}(\bullet|m)$ is the incomplete integral of the first kind, and

$$m = \frac{2P}{P+\omega} \tag{44}$$

[8] This is equivalent to the classical result for the buckling load of a fixed-free strut (see Section 5.2 of O'Reilly and Peters, 2011 and references therein).

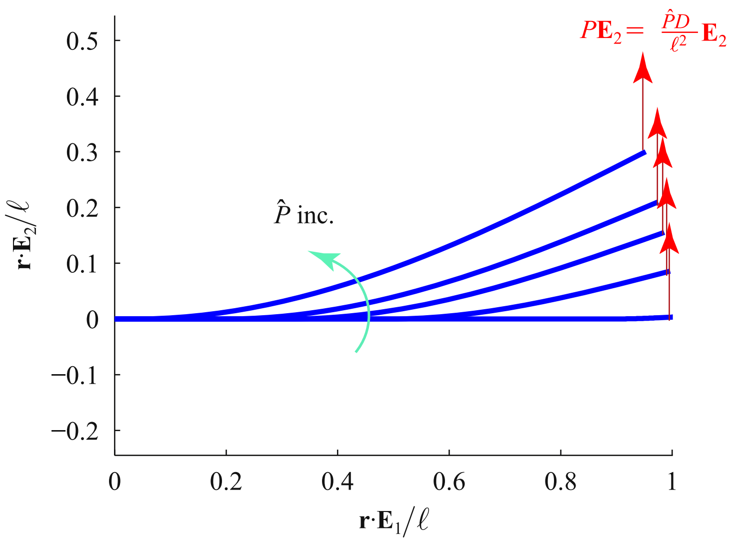

Fig. 6. Configurations of a rod where a force $\mathbf{P} = P\mathbf{E}_2$ is applied to the free end $\xi = \ell$ causing it to peel from a rigid surface. For the five cases shown, the elastica has a dimensionless energy of adhesion $\hat{\omega} = \omega\ell^2/D = 0.5$ and subject to dimensionless loadings of $\hat{P} = P\ell^2/D = 1.1, 1.3, 1.5, 2.0, 10$.

is the elliptic parameter. The elliptic parameter is defined as $m = k^2$ where k is the modulus. Also, $\mathrm{K}(\mu) = \mathrm{F}(\pi/2|\mu)$ represents the complete elliptic integral of the first kind evaluated for the parameter $\mu = 1/m$ (i.e., $k = \sqrt{1/\mu}$).[9] Representative configurations of the rod for various values of P and a fixed value of ω are shown in Fig. 6.

As regards the stability criterion N1, the Riccati boundary-value problem for $w(\xi)$ where $\xi \in (\gamma, \ell]$ and the inequality (32) are, respectively,

$$\frac{\partial w}{\partial \xi} = -P\sin(\theta^*) + \frac{w^2}{D}, \quad w(\ell) = 0, \quad \Gamma \geq 0, \tag{45}$$

where

$$\Gamma = \frac{2\omega}{D}\left(w(\gamma^+) - \left(\sqrt{\frac{D}{2\omega}}\right)P\right). \tag{46}$$

In writing the inequality $(45)_3$, we used the adhesion boundary condition $\theta^{*'}(\gamma^+) = \sqrt{2\omega/D}$.

5.2.1. *Results*

To proceed with analysis of this problem, it is convenient to introduce the dimensionless quantities

$$\hat{\xi} = \frac{\xi}{\ell}, \quad \hat{w} = \frac{w\ell}{D}, \quad \hat{\gamma} = \frac{\gamma}{\ell}, \quad \hat{P} = \frac{P\ell^2}{D}, \quad \hat{\omega} = \frac{\omega\ell^2}{D}. \tag{47}$$

The curves in Fig. 6 represent the shape of the elastica as it is peeled from the substrate. In these calculations, the elastica has a dimensionless energy of adhesion $\hat{\omega} = 0.5$ and is subject to a range of dimensionless loads $\hat{P} = 1.1, 1.3, 1.5, 2, 10$. We observe that greater force leads to less contact. That is, a smaller moment $\mathbf{M}_\gamma$ is required to maintain the boundary condition $(41)_5$.

Examination of the stability criterion N1 leads to the conclusion that peeling is mechanically unstable. To see this, we first calculate $\hat{\gamma}^*$ for prescribed values of $\hat{P}$ and $\hat{\omega}$ and then evaluate the dimensionalized form of the boundary-value problem $(41)_{2-4}$ and $(45)_{1,2}$ over the domain $(\hat{\gamma}^*, 1]$. Representative solutions for $w(\xi)$ can be seen in Fig. 7. Lastly, we substitute the solution for $\hat{w}(\hat{\gamma}^*)$ into $(45)_3$ and determine whether the inequality $\Gamma \geq 0$ is satisfied.

Numerical evaluations of Γ are performed and the results plotted in Fig. 8. Each curve terminates at a minimum value $\hat{P}$, which corresponds to a solution of $\hat{\gamma}^* = 0$. From the figure it is evident that $\Gamma < 0$ for all statically admissible combinations of $\hat{P}$ and $\hat{\omega}$. Clearly, in this case the equilibrium configuration of the rod is stable (to buckling), but, as $\Gamma < 0$, the adhesive is insufficient to maintain the adhesive bond. In summary, the condition N1 is not satisfied and we conclude that peeling is unstable.

5.2.2. *The stability criterion featuring Π_c*

We now turn to the stability criterion featuring Π_c (i.e., (34)). To make the analysis more transparent we restrict attention to the case where θ^* is small. Thus, we start by solving the balance law (from $(41)_2$) $D\theta^{*''} = -P$ with the boundary

[9] The condition $\gamma^* \geq 0$ requires $m \geq 1$ (i.e., $\hat{P} \geq \hat{\omega}$). For numerical packages that require the elliptic parameter m to be between 0 and 1, we make use of the identities $\mathrm{am}(v|m) = \sin^{-1}(\sqrt{\mu}\,\mathrm{sn}(v\sqrt{m}|\mu))$ and $\mathrm{F}(\varphi|m) = \sqrt{\mu}\mathrm{F}(\psi|\mu)$, where $\sin\psi = \sqrt{m}\sin\varphi$ (Byrd and Friedman, 1971). The sine amplitude is defined as $\mathrm{sn}(v|m) = \sin(\mathrm{am}(v|m))$.

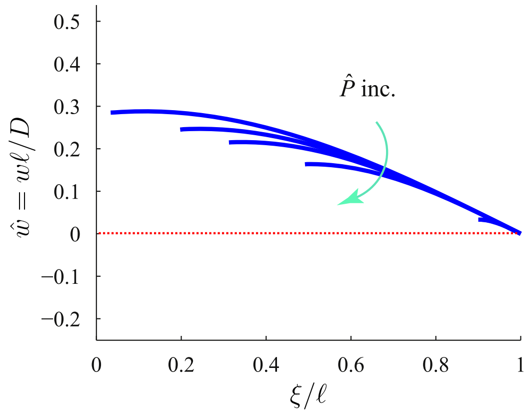

Fig. 7. The normalized solution $\hat{w} = w\ell/D$ to the Riccati boundary-value problem $(45)_{1,2}$ for $\hat{\omega} = \omega\ell^2/D = 0.5$ and values of $\hat{P} = P\ell^2/D = 1.1, 1.3, 1.5, 2, 10$.

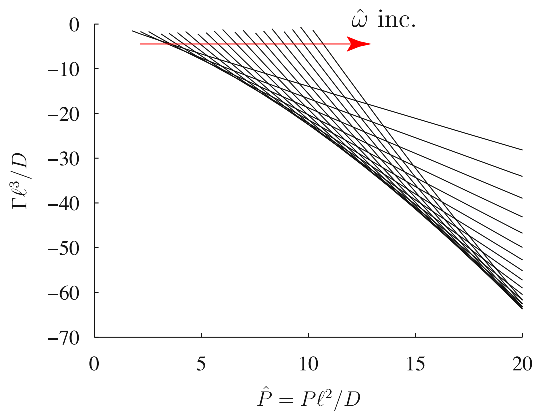

Fig. 8. The function Γ defined by (46) evaluated for prescribed values of $\hat{P}$ and $\hat{\omega} = 1.0, 1.5, 2.0, 2.5, 3.0, \ldots, 10$. The arrow in the figure indicates increasing values of $\hat{\omega}$ and the dimensionless values $\Gamma\ell^3/D$ of Γ are shown.

conditions $\theta^*(\gamma^+) = \theta^{*'}(\ell) = 0$

$$\theta^* = \frac{P}{2D}(2\ell - \gamma - \xi)(\xi - \gamma). \tag{48}$$

This solution is substituted into the expression for the potential energy Π in (8) and Π_c is computed

$$\Pi_c = -\frac{P^2}{6D}(\ell - \gamma)^3 - \omega\gamma. \tag{49}$$

Solving $d\Pi_c/d\gamma = 0$ for $\gamma = \gamma^*$ yields a solution which is equivalent to (18)

$$\gamma^* = \ell - \frac{\sqrt{2\omega D}}{P}. \tag{50}$$

After taking the second derivative of Π_c with respect to γ we find that

$$\left.\frac{d^2\Pi_c}{d\gamma^2}\right|_{\gamma = \gamma^*} = -P\sqrt{\frac{2\omega}{D}}. \tag{51}$$

Clearly $d^2\Pi_c/d\gamma^2|_{\gamma = \gamma^*} < 0$ when $P > 0$, and so the peeling configuration would be classified as unstable when $P > 0$. This is in agreement with our earlier conclusions for the nonlinear boundary-value problem.

5.3. *Shear adhesion of an elastic rod to a rigid flat surface*

As our third, and final example, we consider the case where a rod is bonded to a substrate and subject to applied shearing force. The system is shown schematically in Fig. 1 with $\mathbf{P} = V\mathbf{E}_1$ and $\theta(\ell) = \pi/2$. Motivated by gecko-inspired adhesives this example was also discussed recently in Majidi (2009) and relates to the shear-controlled adhesion of vertically aligned micro- and nanofiber arrays (Lee et al., 2008; Qu et al., 2008).

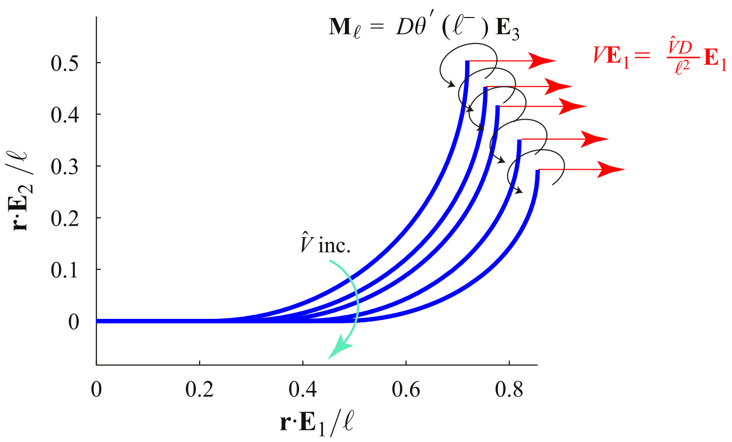

Fig. 9. Configurations of a rod whose free-end is oriented at 90° and is subject to a load $\mathbf{P}=V\mathbf{E}_1$. For the configurations shown, the elastica has a dimensionless energy of adhesion $\hat{\omega}=\omega\ell^2/D=2.0$ and is subject to a range of dimensionless loads $\hat{V}=V\ell^2/D=0,1,2,5,10$ and a reaction moment $\mathbf{M}_\ell=D\theta'(\ell^-)\mathbf{E}_3$ at $\xi=\ell$.

From the balance laws (6) and jump conditions (7) we conclude that

$$\mathbf{M}_\ell = D\theta'(\ell^-)\mathbf{E}_3, \quad \mathbf{n}(\xi)=\begin{cases} V\left(\frac{\xi}{\gamma}\right)\mathbf{E}_1 & \xi\in[0,\gamma), \\ V\mathbf{E}_1 & \xi\in(\gamma,\ell]. \end{cases} \tag{52}$$

Here, $\mathbf{M}_\ell$ is the reaction moment at $\xi=\ell$ which ensures that $\theta(\ell)=\pi/2$ and we are assuming that the applied force is uniformly resisted by the adhesive layer (cf. Figs. 3 and 9).

The shape of the elastic rod can be determined from the following boundary-value problem

$$D\theta^{*''}-V\sin(\theta^*)=0,$$

$$\theta^*(\gamma^+)=0, \quad \theta^*(\ell)=\frac{\pi}{2}, \quad \frac{D}{2}(\theta^{*'}(\gamma^+))^2=\omega. \tag{53}$$

As with the previous peeling problem, γ is a function of V and the solution to the boundary-value problem $(53)_{2-5}$ is

$$\theta^*=\pi-2\mathrm{am}(v|m),$$

$$\gamma^*=\ell-\left\{\mathrm{K}(m)-\mathrm{F}\left(\frac{\pi}{4}\middle|m\right)\right\}\sqrt{\frac{2D}{2V+\omega}}, \tag{54}$$

where

$$v=\mathrm{K}(m)-(\xi-\gamma^*)\sqrt{\frac{2V+\omega}{2D}}, \tag{55}$$

and

$$m=\frac{2V}{2V+\omega}. \tag{56}$$

A detailed derivation is presented in Appendix C.

Paralleling the developments in Section 5.2 for the stability criterion N1, the Riccati boundary-value problem for $w(\xi)$ where $\xi\in(\gamma,\ell]$ and the inequality (32) are, respectively,

$$\frac{\partial w}{\partial \xi}=-V\cos(\theta^*)+\frac{w^2}{D}, \quad w(\ell)=0, \quad w(\gamma^+)\geq 0. \tag{57}$$

The simplicity of the inequality $(57)_3$ can be attributed to the fact that $V\sin(\theta^*(\gamma^+))=0$.

5.3.1. Results

The curves in Fig. 9 illustrate representative shapes of the elastica at static equilibrium. As expected, a greater shear load leads to greater contact between the elastica and substrate. As illustrated by the results shown in Figs. 10 and 11, we find that $\hat{w}(\hat{\gamma}^+)>0$ where $\hat{w}=w\ell/D$ for all combinations of $\hat{V}$ and $\hat{\omega}$. Therefore, the system satisfies the necessary condition for stability. Although we cannot conclude stability, this result is consistent with experimental observations of adhesion under shear loading for an array of vertically aligned polypropylene microfibers (Lee et al., 2008) and multi-walled carbon nanotubes (Qu et al., 2008).

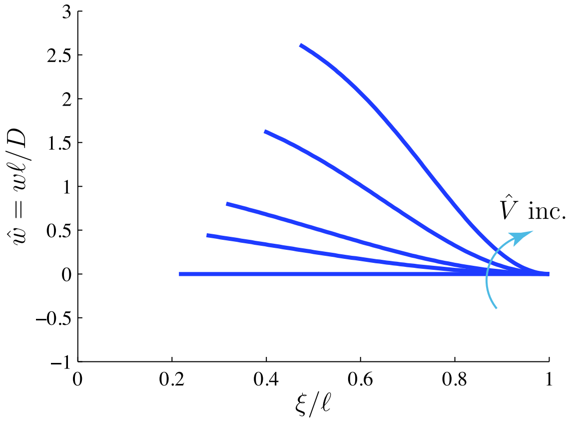

Fig. 10. The normalized solution $\hat{w} = w\ell/D$ to the Riccati boundary-value problem $(57)_{1,2}$ for $\hat{\omega} = \omega\ell^2/D = 0.5$ and a range of values of $\hat{V} = V\ell^2/D = 0, 1, 2, 5, 10$.

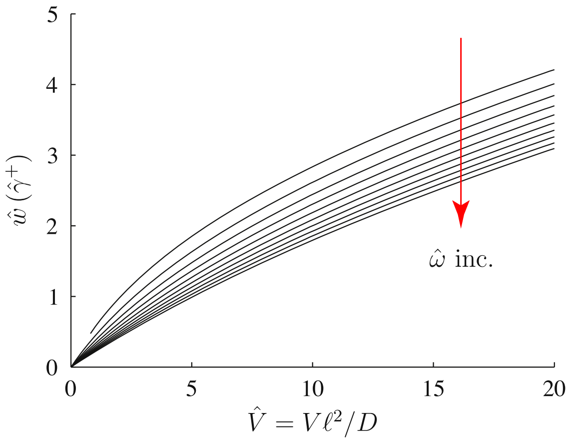

Fig. 11. The boundary value $\hat{w}(\hat{\gamma}^+) = (w(\gamma^+)\ell)/D$ (cf. $(57)_3$) evaluated for prescribed values of $\hat{V}$ and $\hat{\omega} = 1, 2, 3, \ldots, 10$.

5.3.2. The stability criterion featuring Π_c

We now parallel the developments in Section 5.2.2 and present an alternative stability analysis which mimics classical treatments. For the problem at hand, we cannot restrict attention to small values of θ. To proceed, we substitute the result $(54)_1$ for θ^* into the expression for the potential energy Π (cf. (8)). Omitting details, we find an expression for Π which we again denote by Π_c

$$\Pi_c = \Pi_c(\gamma, V, \omega, D, \ell). \tag{58}$$

Using the dimensionless variables (47), we can express this function as

$$\hat{\Pi}_c = \frac{\Pi_c \ell}{D} = \hat{\Pi}_c(\hat{\gamma}, \hat{V}, \hat{\omega}). \tag{59}$$

Taking the partial derivative of $\hat{\Pi}_c$ with respect to $\hat{\gamma}$ and setting the result to 0 we recover an equation which can be solved for $\hat{\gamma}^*$ (see $(54)_2$). As can be seen from Fig. 12, for each value of $\hat{V}$ and $\hat{\omega}$ a unique minimizing $\hat{\gamma}^*$ exists. As expected, this result is consistent with the analysis based on the criterion N1.

6. Concluding remarks

Adhesion and buckling instabilities govern elastic deformation and contact between an elastica and a rigid halfspace. We have simultaneously addressed both instability modes with a comprehensive analysis that uniquely combines stationary principles and the calculus of variations with classical transformations by Legendre and Jacobi. In particular, we showed that stability requires the existence of solutions to a Riccati equation which satisfy an inequality. This inequality is intimately related to the adhesion boundary condition and supplements the usual existence result that is needed when buckling instabilities alone are being considered. We have also shown how our treatment is an improvement to classical treatments where the potential energy is expressed as a function of γ and the system parameters.

Several open problems remain for the adhered rod problems under loading control. First, a more general sufficiency condition for stability that does not possess the boundary limitations associated with S1 would be desirable. On a more applied level, we note that while the peeling solution is unstable, it would be of interest to examine the stability of adhered

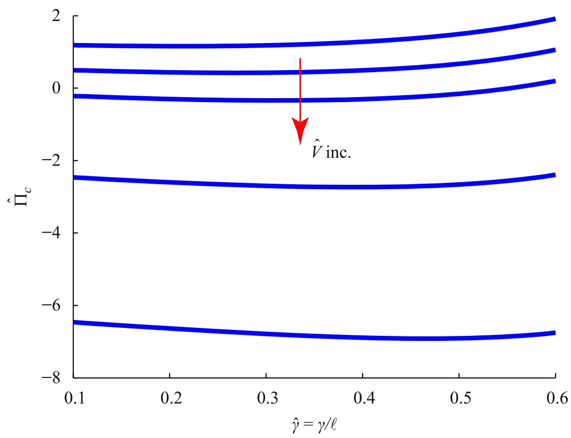

Fig. 12. Representative examples of the function $\hat{\Pi}_c(\hat{\gamma},\hat{V},\hat{\omega})$ (cf. (59)) for various values of $\hat{V}=0,1,2,5$, and 10, and $\hat{\omega}=2$.

rod when the terminal load $\mathbf{P}=V(\cos(\chi)\mathbf{E}_1+\sin(\chi)\mathbf{E}_2)$ where χ is a constant. For given values of V, ω, ℓ and D, we suspect that a critical value of χ can be found above which peeling dominates and the equilibrium configuration becomes unstable. A further class of open problems is the development of stability criteria to situations where several limbs of a branched tree-like structure are in adhesive contact (see Pugno, 2011 and references therein for examples).

The stability criteria we have developed are restricted to problems where the external loading is controlled and the adhesion is modeled in a very simple manner. It would be of clear benefit to extend our analysis to problems featuring displacement control. Traditionally, this has entailed introducing isoperimetric constraints which we have yet to consider. With regards to the adhesive, our model does not include rate effects such as viscoelasticity. In addition the instability mechanism it predicts is restricted to either the rod buckling or the adhesive being of insufficient strength. We remark that the easiest example to see the latter effect is in the peeling problem discussed in Section 5.2.

Acknowledgments

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Appendix A. Proof of the condition N1

The condition N1 presented in Section 4.3 is a necessary condition for the second variation J to be non-negative. Our proof of this criterion closely follows Bliss' well-known proof of Jacobi's necessary condition for the classical variational problem with fixed endpoints (see, e.g., Theorem 4 in Bliss, 1916 or Theorem 2.10 in the textbook Ewing, 1969).

First, we restrict our attention to variations where γ is unchanged: $\mu=0$. As $\mu=0$, we are interested in finding piecewise smooth functions $\chi(\xi)$ which minimize $J=\hat{J}$ subject to appropriate boundary conditions on $\chi(\xi)$

$$\hat{J}=\int_{\gamma}^{\ell}(D\chi'\chi'+P\chi^2)\,d\xi,\quad \chi(\gamma^+)=0,\quad \chi'(\ell)=0. \tag{A.1}$$

This problem is known as the accessory variational problem and the appropriate necessary conditions for $\chi(\xi)$ are that it satisfies a Jacobi differential equation (cf. (29)) and, at points $\xi=\xi_d$ where χ' has a discontinuity, that the Weierstrass–Erdmann corner conditions are satisfied

$$D\chi''-P\chi=0,\quad [\![D\chi']\!]_{\xi_d}=0,\quad [\![P\chi^2-D\chi'\chi']\!]_{\xi_d}=0. \tag{A.2}$$

The proof of the criterion is established by contradiction. We assume that $\hat{J}\geq 0$, but suppose that for some $\xi=\xi_c\in(\gamma,\ell)$ the solution $w(\xi)$ to the Riccati equation (25) becomes unbounded. Thus, the corresponding solution u to (29) and (30) is such that $u(\xi_c)=0$ (i.e., $\xi=\xi_c$ is a conjugate point to $\xi=\ell$ in the interval (γ^*,ℓ)).[10] It follows that we can construct a variation

$$\overline{\eta}(\xi)=\begin{cases}0\ \forall\xi\in[\gamma,\xi_c],\\ u(\xi)\ \forall\xi\in(\xi_c,\ell].\end{cases} \tag{A.3}$$

[10] An example of this situation is shown in Fig. 4(b).

Substituting $\overline{\eta}$ for χ in the expression (A.1)$_1$ for $\hat{J}$ and performing an integration by parts, we find that $\hat{J}=0$

$$\hat{J}=\int_{\gamma}^{\ell}(D\overline{\eta}'\overline{\eta}'+P\overline{\eta}^2)\,d\xi=0. \tag{A.4}$$

However, $\overline{\eta}'$ has a discontinuity at $\xi=\xi_c$ and this violates the necessary Weierstrass–Erdmann corner condition $[\![D\overline{\eta}']\!]_{\xi_c}=0$ for an extremum (cf. (A.2)$_2$). Thus, we have constructed a function $\overline{\eta}$ which allows J to achieve its supposed minimum value of 0, but which does not satisfy the necessary condition for an extremizer. We conclude that the minimum value of J is less than 0 and this is the desired contradiction. In conclusion, if $\{\theta^*(\xi),\gamma^*\}$ minimizes Π, then the solution to (25) with $w(\ell)=0$ cannot become unbounded.

To examine the sign of J_2 we note that $\eta(\ell)$ is arbitrary and this motivated our earlier choice of the initial condition $w(\ell)=0$. Appealing to the compatibility conditions (22) and the balance law (13), a more convenient expression for J_2 can be established

$$J_2=\eta^2(\gamma^+)w(\gamma^+)+\theta^{*'}(\gamma^+)P_1\mu^2=\theta^{*'}(\gamma^+)(\theta^{*'}(\gamma^+)w(\gamma^+)+P_1)\mu^2. \tag{A.5}$$

The necessary condition for $J_2\geq 0$ that is discussed in the condition N1 easily follows from (A.5).

Appendix B. Derivation of Eq. (42)

We begin our derivation of (42) by rearranging the terms in (41)$_2$

$$\frac{d\theta'}{d\xi}=-\frac{P}{D}\cos(\theta). \tag{B.1}$$

The $*$ ornamenting θ is dropped in this appendix. A standard integration yields

$$\frac{1}{2}(\theta')^2=C-\frac{P}{D}\sin(\theta), \tag{B.2}$$

where C is a constant of integration. Evaluating (B.2) at $\xi=\gamma$ and applying the adhesion boundary conditions (41)$_3$ and (41)$_5$ shows that $C=\omega/D$. Next, rearranging terms and introducing the transformation $\theta=2\varphi-\pi/2$ leads to

$$\sqrt{\frac{2D}{P+\omega}}\frac{d\varphi}{d\xi}=\sqrt{1-m\sin^2(\varphi)}, \tag{B.3}$$

where $m=2P/(P+\omega)$. In order to arrive at (B.3), we made use of the trigonometric identities $\sin(\varphi-\pi/2)=-\cos(\varphi)$ and $\cos(2\varphi)=1-2\sin^2(\varphi)$.

Integrating (B.3) over the domain $\Omega=(\gamma,\xi]$ yields

$$\int_{\Omega}\frac{\mathrm{d}\varphi}{\sqrt{1-m\sin^2(\varphi)}}=\int_{\Omega}\sqrt{\frac{P+\omega}{2D}}\,d\xi. \tag{B.4}$$

Noting that $\theta(\gamma)=0$, it follows that $\varphi(\gamma)=\pi/4$. After integrating (B.4), we find that

$$\mathrm{F}(\varphi|m)-\mathrm{F}\left(\frac{\pi}{4}\middle|m\right)=(\xi-\gamma)\sqrt{\frac{P+\omega}{2D}}. \tag{B.5}$$

The Jacobi amplitude φ is determined by simply inverting the incomplete elliptic integral of the first kind $\mathrm{F}(\phi|m)$ to obtain

$$\varphi=\mathrm{am}(\nu|m),\quad \nu=(\xi-\gamma)\sqrt{\frac{P+\omega}{2D}}+\mathrm{F}\left(\frac{\pi}{4}\middle|m\right). \tag{B.6}$$

Substituting the solution (B.6) for φ into the expression $\theta=2\varphi-\pi/2$ yields the result

$$\theta=2\mathrm{am}(\nu|m)-\frac{\pi}{2}, \tag{B.7}$$

which corresponds to (42)$_1$.

The contact length γ is determined by applying the boundary condition (41)$_4$ to (B.3). Noting that $\theta'(\ell)=0$ it follows from (B.3) that $\sin(\varphi)=\sqrt{\mu}$, where $\mu=1/m$. Next, we invoke the Jacobi identity $\mathrm{sn}(\nu|m)=\sqrt{\mu}\,\mathrm{sn}(\nu\sqrt{m}|\mu)$. Together, $\mathrm{sn}(\nu|m)=\sqrt{\mu}$ and $\mathrm{sn}(\nu|m)=\sqrt{\mu}\,\mathrm{sn}(\nu\sqrt{m}|\mu)$ imply that $\mathrm{sn}(\nu\sqrt{m}|\mu)=1$ at $\xi=\ell$. This implies that $\nu\sqrt{m}=\mathrm{K}(\mu)$ and so evaluating ν at ℓ leads to the identity

$$(\ell-\gamma)\sqrt{\frac{P+\omega}{2D}}+\mathrm{F}\left(\frac{\pi}{4}\middle|m\right)=\frac{1}{\sqrt{m}}\mathrm{K}(\mu). \tag{B.8}$$

Rearranging terms to solve for γ yields (42)$_2$.

Appendix C. Derivation of Eq. (54)

As in Appendix B, we begin by performing a separation of variables on the ordinary differential equation $\theta'' = (V/D)\sin(\theta)$ and integrating to obtain

$$\frac{1}{2}(\theta')^2 = C - \frac{V}{D}\cos(\theta). \tag{C.1}$$

The $*$ ornamenting θ is also dropped in this appendix and the constant of integration C is determined by applying the boundary conditions $(53)_{2,4}$: $C=(V+\omega)/D$. Next, introducing the transformation $\theta = \pi - 2\varphi$ and rearranging terms yields

$$-\sqrt{\frac{2D}{2V+\omega}}\frac{d\varphi}{d\xi} = \sqrt{1-m\sin^2(\varphi)}, \tag{C.2}$$

where $m = 2V/(2V+\omega)$.

We now perform a separation of variables and integrate over the domain $\Omega = (\gamma,\xi]$

$$\int_\Omega \frac{d\varphi}{\sqrt{1-m\sin^2(\varphi)}} = \int_\Omega -\sqrt{\frac{2V+\omega}{2D}}\,d\xi. \tag{C.3}$$

Noting that $\varphi(\gamma^+) = \pi/2$, it follows that

$$\mathrm{F}(\varphi|m) - \mathrm{K}(m) = -(\xi-\gamma)\sqrt{\frac{2V+\omega}{2D}}. \tag{C.4}$$

Inverting (C.4) leads to the following solution for θ

$$\theta = \pi - 2\,\mathrm{am}(\nu|m), \tag{C.5}$$

where

$$\nu = \mathrm{K}(m) - (\xi-\gamma)\sqrt{\frac{2V+\omega}{2D}}. \tag{C.6}$$

This is the same result presented in $(54)_1$.

According to $(53)_3$, $\theta(\ell) = \pi/2$, which implies $\varphi(\ell) = \pi/4$. Evaluating (C.4) at $\xi = \ell$ enables us to conclude that

$$\gamma = \ell - \left\{\mathrm{K}(m) - \mathrm{F}\left(\frac{\pi}{4}\middle| m\right)\right\}\sqrt{\frac{2D}{2V+\omega}}. \tag{C.7}$$

This solution for γ corresponds to $(54)_2$.

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