## Carmel Majidi

Princeton Institute for the Science and Technology of Materials, Princeton University, Princeton, NJ 08544 e-mail: cmajidi@princeton.edu

## Kai-tak Wan<sup>1</sup>

Associate Professor Mechanical and Industrial Engineering, Northeastern University, Boston, MA 02115 e-mail: ktwan@coe.neu.edu

# Adhesion Between Thin Cylindrical Shells With Parallel Axes

Energy principles are used to investigate the adhesion of two parallel thin cylindrical shells under external compressive and tensile loads. The total energy of the system is found by adding the strain energy of the deformed cylinder, the potential energy of the external load, and the surface energy of the adhesion interface. The elastic solution is obtained by linear elastic plate theory and a thermodynamic energy balance, and is capable of portraying the measurable quantities of external load, stack height, contact arc length, and deformed profile in the reversible process of loading-adhesion and unloading-delamination. Several worked examples are given as illustrations. A limiting case of adhering identical cylinders is shown to be consistent with recent model constructed by Tang et al. Such results are of particular importance in modeling the aggregation of heterogeneous carbon nanotubes or cylindrical cells, where the contacting microstructures have a different radius and/or bending stiffness. [DOI: 10.1115/1.4000924]

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## **1** 1 Introduction

The adhesion of thin-walled micro- and nanoscale structures 2 **3** governs the functionality of many emerging technologies [1,2]. 4 Fabrication methods in nanotechnology include adhesion-5 controlled manipulation and assembly of thin-walled structures such as carbon nanotubes (CNTs) single fibers and bundles, 6 graphene sheets, and fullerenes [3]. Thin-film/thin-wall adhesion 7 8 also controls the stability and structural integrity of flexible nanoelectronics and microtruss structures, which are subject to stiction 10 and potential collapse under environment induced adhesion (e.g., 11 meniscus formation at high relative humidity) [4,5]. Moreover, **12** there has been growing interest in the role of thin-walled adhesion 13 in biological and pathophysiological systems. Waste water treat-14 ment relies on the adhesion-controlled aggregation of bacteria, 15 and the formation of biofilm [6] and cell-cell adhesion helps form 16 natural and prosthetic tissues [7]. Excessive adhesion causes 17 monocytes to bond to the aorta wall, which eventually obstructs **18** the vessels and leads to atherosclerotic plaques [8], whereas lack **19** of adhesion results in the loss of synaptic contacts and gives rise **20** to Alzheimer disease [9].

21 Developing insights and predictive models for these systems 22 requires an understanding of the mechanics of adhesion between 23 thin-walled structures as a result of intersurface forces such as 24 electrostatic, van der Waals interactions, and meniscus. To achieve mechanical equilibrium, the adhesion energy must balance the 25 26 mechanical energies due to external load and structural deformation [10]. Notwithstanding the many existing and successful solid-27 solid adhesion models, a new theory is needed to explicitly ad-28 29 dress adhesion between thin-walled structures that are dissimilar 30 in stiffness, geometry, and dimension. Here, we consider one par-31 ticular class of geometries: parallel, thin-walled cylinders with 32 dissimilar bending rigidity and radius. The new model has the 33 potential to be extended to other geometries, such as contacting **34** circular plates and thin-walled spheres [11].

an external load on two noninteracting spheres leads to a compres- 37 sive stress within the contact circle. Modifications to include in- 38 terfacial adhesion were later introduced by Johnson-Kendall- 39 Roberts (JKR), Derjaguin-Muller-Toporov (DMT), and Dugdale- 40 Barenblatt-Maugis [10]. In essence, the interfacial attraction 41 modifies the local deformation and introduces a tensile stress 42 around the largely compressive contact circle. Relationships be- 43 tween applied load, contact radius, and approach distance are veri- 44 fied in a wide range of materials and interfaces. The theory is 45 further extended to the adhesion of a solid sphere with a wavy 46 substrate [12,13], a solid cylinder with a planar substrate, and 47 cylinders with parallel axes [14,15]. However, these models are 48 inadequate for thin shells in that the shell conforms to the sub- 49 strate geometry by deforming in plate-bending, membrane- 50 stretching, or mixed bending-stretching mode such that the notion 51 of central compression is excluded. New models are recently de- 52 veloped for freestanding planar circular membranes clamped at 53 the periphery and a planar substrate in the presence of finite range 54 intersurface attraction, though membrane deformation is con- 55 strained to membrane stretching and negligible bending 56 [4, 16-20].57 Thin shell adhesion on a planar substrate has been investigated 58

Virtually all existing adhesion models are based on the Hertz **35** contact theory. Because of geometrical incompatibility, exerting **36** 

extensively with numerical methods. Seifert [21] treated lipid 59 vesicles as shells, developed a mechanical model by balancing the 60 adhesion energy with Helfrich's elastic bending, and constructed a 61 self-consistent theory for bounded and unbounded vesicles. Tang 62 et al. [22] and Glassmaker and Hui [23] constructed an elastic 63 model for two interacting CNTs that was consistent with molecu- 64 lar mechanics simulation. A critical shell radius is found below 65 which the contact remains a line:  $R_{\min} = (k/\gamma)^{1/2}$ , where k is the 66 shell stiffness and  $\gamma$  is the adhesion energy. Adams, Pamp, and 67 Majidi introduced the moment-discontinuity-method to analyze 68 the adhesion of intrinsically curved plates and beams to curved 69 substrates [24,25]. Springman and Bassani [26,27] adopted a nu- 70 merical method to probe a spherical capped shell attracted to a 71 planar substrate via a finite range Lennard-Jones potential, derived 72 the "pull-in" and "pull-off" events, and further extended their 73 model to wavy substrates under coupled chemomechanical inter- 74 actions. 75

<sup>&</sup>lt;sup>1</sup>Corresponding author.

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Fig. 1 Interactions between two cylindrical shells: (a) touching at a line contact without adhesion, (b) compressive deformation with adhesion, and (c) tensile deformation with adhesion

In this paper, we attempt to address the global deformation of 76 77 two elastic cylinders with parallel axis under the following assumptions: (i) Both cylinders are hollow shells with infinite 78 length, (ii) bending is the dominant deformation mode, and (iii) 79 80 the intersurface attraction is effective at intimate contact conform-81 ing to the JKR assumption [28]. A boundary condition is intro-82 duced to represent the discontinuity in bending curvature at the 83 contact edge. This is an extension of the moment-discontinuity-**84** method [26,27] and is derived by minimizing the total potential 85 energy of the system with respect to the width or radius of the **86** contact zone. This boundary condition may also be derived using 87 methods of fracture mechanics such as the J-integral [23,29] and the stress intensity factor [30]. However, in contrast to the current 88 89 analysis, these derivations are beyond the scope of conventional 90 plate and shell theory and require the evaluation of internal stress **91** and strain fields.

#### **92** 2 Model

 Figure 1 shows two cylinders with natural undeformed radii  $R_1$  and  $R_2$  being pressed into contact and then separated. Figure 2 shows the curvilinear coordinates. Upon a compressive force *F*, the cylinders deform to create a finite contact segment of arc length 2*a*. As *F* becomes tensile (negative), adhesion contact re- mains until a critical pull-off load  $F_0$  is reached. A spontaneous separation of the adherends follows that reduces *a* to zero.

100 Let  $s_1$  and  $s_2$  denote the arc lengths of the bottom and top 101 cylinders, respectively, measured from the cylinder poles. Sym-102 metry about the vertical axis requires the left-half of the system to 103 be considered, and analysis is limited to  $L_1 = \pi R_1$  and  $L_2 = \pi R_2$ . 104 Define



Fig. 2 Curvilinear coordinates

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$$\xi_1 = L_1 - a$$
 and  $\xi_2 = L_2 - a$  (1) 10

corresponding to the arc length at which the bottom and top cyl- 106 inders make contact. In their natural configuration, the cylinders 107 are deflected by an angle 108

$$\theta_1 = s_1/R_1$$
 and  $\theta_2 = -s_2/R_2$  (2) 109

with respect to horizontal. Under an applied load, the deflection 110 increases by an angle  $\phi_1$  and  $\phi_2$  such that the final deflection is 111  $\theta_1 + \phi_1$  and  $\theta_2 + \phi_2$ . 112

**2.1 Boundary Conditions.** The angular deformations  $\phi_1$  **113**  $= \phi_1(s_1)$  and  $\phi_2 = \phi_2(s_2)$  and arc length *a* must satisfy boundary **114** conditions that ensure both mirror symmetry about the vertical **115** axis and geometric compatibility between the cylinders along their **116** contact. Noting that  $\theta_1(0) = \theta_2(0) = \theta_1(L_1) = \theta_2(L_2) = 0$ , it follows **117** that in order for symmetry to be preserved, the boundary condi- **118** tions **119** 

$$\phi_1(0) = \phi_2(0) = \phi_1(L_1) = \phi_2(L_2) = 0$$
 (3) 120

must be satisfied. To ensure geometric compatibility and to pre- 121 vent interpenetration of the adhering surfaces, the two cylinders 122 must share the same shape along the length of contact. Referring 123 to Fig. 2(*b*), this requires  $\pi - (\theta_1 + \phi_1)$  to equal  $-(\theta_2 + \phi_2) - \pi$  for 124 all values of  $s_1 \in [\xi_1, L_1]$  and  $s_2 \in [\xi_2, L_2]$ , where  $s_2 = s_1 - \xi_1 + \xi_2$ , 125

$$\theta_1(s_1) + \phi_1(s_1) = 2\pi + \theta_2(s_1 - \xi_1 + \xi_2) + \phi_2(s_1 - \xi_1 + \xi_2), \quad \forall s_1$$
 **126**

$$\in [\xi_1, L_1] \tag{4} 127$$

Lastly, the deformations  $\phi_1$  and  $\phi_2$  must allow the cylinders to **128** form a close loop such that the isoperimetric constraints **129**  $\int_{-1}^{0} \cos(\theta_1 + \phi_1) ds_1 = \int_{-1}^{0} \cos(\theta_2 + \phi_2) ds_2 = 0$  are satisfied. In light of **130** the compatibility condition in Eq. (4), this is equivalent to **131** 

$$\int_{0}^{\xi_{1}} \cos(\theta_{1} + \phi_{1}) ds_{1} = \int_{0}^{\xi_{2}} \cos(\theta_{2} + \phi_{2}) ds_{2}$$
132

$$= -\int_{\xi_1}^{1} \cos(\theta_1 + \phi_1) ds_1$$
 (5)  
**133**

134

At this point it is convenient to define

$$\phi_a = \{\phi_1 : s_1 \in [\xi_1, L_1]\} \tag{6}$$

This allows deformation to be represented by three independent 136 functions  $\phi_1$ ,  $\phi_2$ , and  $\phi_a$  on the domains  $[0, \xi_1]$ ,  $[0, \xi_2]$ , and 137  $[\xi_1, L_1]$ , respectively. By introducing  $\phi_a$ , the boundary conditions 138 reduce to 139

$$\phi_1(0) = \phi_2(0) = \phi_a(L_1) = 0 \tag{7}$$
 140

$$\phi_a(\xi_1) = \phi_1(\xi_1) = 2\pi + \phi_2(\xi_2) + \theta_2(\xi_2) - \theta_1(\xi_1)$$
(8) 141

$$\int_{0}^{\xi_{1}} \cos(\theta_{1} + \phi_{1}) ds_{1} = \int_{0}^{\xi_{2}} \cos(\theta_{2} + \phi_{2}) ds_{2}$$
142

$$= -\int_{\xi_1}^{\xi_1} \cos(\theta_1 + \phi_a) ds_1$$
 (9) (44)

It is important to note that these conditions explicitly prevent interpenetration of the cylinders *only* along the contact zone  $(s_1 \ 145 \in [\xi_1, L_1])$ .

**2.2 Energy Functional.** The cylindrical walls are treated as 147 inextensible *elastica*. Hence, extension and shear strains are ig- 148 nored and the elastic strain energy is limited to bending. Let  $k_1$  149 and  $k_2$  denote the dimensionless flexural rigidity of the bottom and 150 top cylinders, respectively, where both  $k_i$  are normalized with re- 151 spect to the flexural rigidity of cylinder 1,  $D_1 = E_1 h_1 / 12(1 - v_1^2)$ , 152 with  $E_1$  the elastic modulus,  $v_1$  Poisson's ratio, and  $h_1$  the wall 153 thickness. The total elastic strain energy of the system  $\Gamma$  can be 154

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155 decomposed into the segments corresponding to the domains **156**  $[0, \xi_1], [0, \xi_2], \text{ and } [\xi_1, L_1] \text{ as follows:}$ 

157  

$$\Gamma_{1} = \int_{0}^{\xi_{1}} \frac{1}{2}k_{1}\phi_{1,1}^{2}ds_{1}, \quad \Gamma_{2} = \int_{0}^{\xi_{2}} \frac{1}{2}k_{2}\phi_{2,2}^{2}ds_{2}$$

$$\Gamma_{3} = \int_{\xi_{1}}^{L_{1}} \left\{ \frac{1}{2}k_{1}\phi_{a,1}^{2} + \frac{1}{2}k_{2}\left(\phi_{a,1} + \frac{1}{R_{1}} + \frac{1}{R_{2}}\right)^{2} \right\} ds_{1}$$

158

 where  $\phi_{i,j} = d\phi_i/ds_j$ . The total potential energy of the system  $\Pi$  is 160 computed by combining these elastic strain energies with the work  $U_f$  of the external load F, the virtual work  $U_{\lambda}$  of the isoperimetric constraints in Eq. (9), and the work of adhesion  $W = \gamma a$ . That is,

163 
$$\Pi = \Gamma_1 + \Gamma_2 + \Gamma_3 + U_f + U_\lambda - W \tag{11}$$

164 where

$$U_F = \int_0^{\xi_1} F \sin(\theta_1 + \phi_1) ds_1 - \int_0^{\xi_2} F \sin(\theta_2 + \phi_2) ds_2 \quad (12)$$

165 166 and

167  
$$U_{\lambda} = \int_{0}^{\xi_{1}} \lambda_{1} \cos(\theta_{1} + \phi_{1}) ds_{1} + \int_{0}^{\xi_{2}} \lambda_{2} \cos(\theta_{2} + \phi_{2}) ds_{2} + \int_{0}^{L_{1}} (\lambda_{1} + \lambda_{2}) \cos(\theta_{1} + \phi_{a}) ds_{1}$$
(13)

168

 The Lagrangian multipliers  $\lambda_1$  and  $\lambda_2$  in Eq. (13) are unknown 170 constants and correspond to the internal "hoop" stress at the points  $s_1 = s_2 = 0$ . The total potential energy of the system may be ex-pressed by the functional

173 
$$\Pi = \int_{0}^{\xi_{1}} \left\{ \frac{1}{2} k_{1} \phi_{1,1}^{2} + F \sin(\theta_{1} + \phi_{1}) + \lambda_{1} \cos(\theta_{1} + \phi_{1}) \right\} ds_{1}$$

$$+ \int_{0}^{\xi_{2}} \left\{ \frac{1}{2} k_{1} \phi_{1,1}^{2} - F \sin(\theta_{1} + \phi_{1}) + \lambda_{1} \cos(\theta_{1} + \phi_{1}) \right\} ds_{1}$$

+ 
$$\int_{0} \left\{ \frac{1}{2} k_2 \phi_{2,2}^2 - F \sin(\theta_2 + \phi_2) + \lambda_2 \cos(\theta_2 + \phi_2) \right\} ds_2$$

$$+ \int_{\xi_1}^{L_1} \left\{ \frac{1}{2} (k_1 + k_2) \phi_{a,1}^2 + k_2 \phi_{a,1} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + \frac{1}{2} k_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^2 \right\}$$

176 
$$+ (\lambda_1 + \lambda_2)\cos(\theta_1 + \phi_a) - \gamma \bigg| ds_1$$
(14)

#### 177 3 Analysis

At equilibrium, the energy functional  $\Pi$  must be stationary with 178 **179** respect to kinematically admissible variations of the form

$$\phi_1 = \phi_1^* + \delta\phi_1, \quad \phi_2 = \phi_2^* + \delta\phi_2, \quad \phi_a = \phi_a^* + \delta\phi_a, \quad a = a^* + \delta a$$
**180**
(15)

**181** Here,  $\chi^*$  denotes the value of  $\chi$  at equilibrium and  $\delta \chi$  is an arbi-**182** trary but infinitesimally small variation from  $\chi^*$ . In the subsequent 183 analysis, it is convenient to define the Lagrangian densities

**184** 
$$\Lambda_1 = \frac{1}{2}k_1\phi_{1,1}^2 + F\sin(\theta_1 + \phi_1) + \lambda_1\cos(\theta_1 + \phi_1)$$

**185** 
$$\Lambda_2 = \frac{1}{2}k_2\phi_{2,2}^2 - F\sin(\theta_2 + \phi_2) + \lambda_2\cos(\theta_2 + \phi_2)$$

**186** 
$$\Lambda_a = \frac{1}{2}(k_1 + k_2)\phi_{a,1}^2 + k_2\phi_{a,1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{2}k_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^2$$

187 
$$+ (\lambda_1 + \lambda_2)\cos(\theta_1 + \phi_a) - \gamma$$
(16)

**3.1 Balance Laws.** Let  $\delta \Pi_{\phi}$  denote the variation in  $\Pi$  in-188 189 duced by the first three variations in Eq. (15). Employing the calculus of variations and noting that the variations must be kine- 190 matically admissible, it is straightforward to show that  $\delta \Pi_{\phi}$  van- 191 ishes if and only if the balance laws 192

$$\frac{\partial \Lambda_1}{\partial \phi_1} - \frac{d}{ds_1} \left( \frac{\partial \Lambda_1}{\partial \phi_{1,1}} \right) = 0, \quad \frac{\partial \Lambda_2}{\partial \phi_2} - \frac{d}{ds_2} \left( \frac{\partial \Lambda_2}{\partial \phi_{2,2}} \right) = 0,$$
193

$$\frac{\partial \Lambda_a}{\partial \phi_a} - \frac{d}{ds_1} \left( \frac{\partial \Lambda_a}{\partial \phi_{a,1}} \right) = 0 \tag{17}$$

and natural boundary condition

(10)

$$\left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}}\right)_{s_1=\xi_1} + \left(\frac{\partial \Lambda_2}{\partial \phi_{2,2}}\right)_{s_2=\xi_2} - \left(\frac{\partial \Lambda_a}{\partial \phi_{a,1}}\right)_{s_1=\xi_1} = 0$$
(18)  
196

are satisfied (see Appendix A for derivation). Equation (17) cor- 197 responds to the differential form of the moment balance along the 198 segments  $s_1 \in [0, \xi_1]$ ,  $s_2 \in [0, \xi_2]$ , and  $s_1 \in [\xi_1, L_1]$ , respectively, **199** while Eq. (18) corresponds to the moment balance at the edge of 200 the interface  $(s_1 = \xi_1)$ .

Substituting the Lagrangian densities into Eq. (17) results in a 202 system of three second-order ordinary differential equations. Solv- 203 ing these will introduce six constants of integration 204  $(c_1, c_2, \dots, c_6)$ , resulting in altogether nine unknowns: 205  $a, \lambda_1, \lambda_2, c_1, c_2, \ldots, c_6$ . However, so far, we have presented only 206 eight linearly independent equations: the five boundary conditions 207 in Eqs. (7) and (8), the two isoperimetric constraints in Eq. (9), 208 and moment balance (18) at  $s_1 = \xi_1$  and  $s_2 = \xi_2$ . In order to calculate 209 the unknown constants, a ninth linearly independent equation is 210 required. This is furnished by the fourth variation in Eq. (15) and 211 is presented in Sec. 3.2. 212

3.2 Jump Condition. The fourth variation in Eq. (15) results 213 in a variation of the potential energy that has the form  $\delta \Pi_a$  214  $=(d\Pi/da)\delta a$ . Since  $\delta a$  is arbitrary,  $\delta \Pi_a$  vanishes if and only if **215**  $d\Pi/da=0$ . Employing Leibniz' integration rule, the chain rule, 216 the balance laws in Eq. (17), and the natural boundary condition 217 in Eq. (18), it follows that  $d\Pi/da=0$  reduces to 218

$$\begin{aligned} (\Lambda_{a})_{s_{1}=\xi_{1}} - (\Lambda_{1})_{s_{1}=\xi_{1}} - (\Lambda_{2})_{s_{2}=\xi_{2}} + \left(\frac{\partial\Lambda_{1}}{\partial\phi_{1,1}}\right)_{s_{1}=\xi_{1}} \{\phi_{1,1}(\xi_{1}) - \phi_{a,1}(\xi_{1})\} \\ + \left(\frac{\partial\Lambda_{2}}{\partial\phi_{2,2}}\right)_{s_{2}=\xi_{2}} \left\{\phi_{2,2}(\xi_{2}) - \phi_{a,1}(\xi_{1}) - \frac{1}{R_{1}} - \frac{1}{R_{2}}\right\} = 0 \end{aligned} \tag{19}$$

Details of the derivation are provided in Appendix B. Jump con- 221 dition (19) provides the ninth equation necessary to complete the 222 system of linear equations needed to solve for the nine unknown 223 constants:  $a, \lambda_1, \lambda_2, c_1, c_2, \dots, c_6$ . Physically, Eq. (19) corresponds 224 to the balance of the work of adhesion with the elastic energy 225 release rate associated with variations of the arc length a from its 226 value at equilibrium. 227

3.3 Solution. The governing equations are derived by substi- 228 tuting the expressions for  $\Lambda_1$ ,  $\Lambda_2$ , and  $\Lambda_a$  into the above equa- 229 tions. A solution can easily be obtained by linearizing for small  $\phi_1$  230 and  $\phi_2$ . This yields the following set of governing equations (see 231 Appendix C): 232

$$k_1 \phi_{1,11} = F \cos(\theta_1) - \lambda_1 \sin(\theta_1)$$
 (20) 233

$$k_2 \phi_{2,22} = -F \cos(\theta_2) - \lambda_2 \sin(\theta_2)$$
 (21) 234

$$(k_1 + k_2)\phi_{a,11} = -(\lambda_1 + \lambda_2)\sin(\theta_1)$$
 (22) 235

Also, natural boundary condition (18) and jump condition (19) 236 imply 237

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195

$$k_{1}\{\phi_{1,1}(\xi_{1}) - \phi_{a,1}(\xi_{1})\}k_{2}\left\{\phi_{2,2}(\xi_{2}) - \phi_{a,1}(\xi_{1}) - \frac{1}{R_{1}} - \frac{1}{R_{2}}\right\} = 0$$
(23)

(24)

239 and

240 
$$\frac{1}{2}k_{1}\{\phi_{1,1}(\xi_{1}) - \phi_{a,1}(\xi_{1})\}^{2} + \frac{1}{2}k_{2}\left\{\phi_{2,2}(\xi_{2}) - \phi_{a,1}(\xi_{1}) - \frac{1}{R_{1}} - \frac{1}{R_{2}}\right\}$$
241 =  $\gamma$ 

242 respectively. As before, the kinematic boundary conditions are

**243** 
$$\phi_1(0) = \phi_2(0) = \phi_a(L_1) = 0$$
 (25)

**244** 
$$\phi_a(\xi_1) = \phi_1(\xi_1) = 2\pi + \phi_2(\xi_2) + \theta_2(\xi_2) - \theta_1(\xi_1)$$
 (26)

245 Lastly, linearization reduces the isoperimetric constraints to

246 
$$\int_{0}^{\xi_{1}} \{\cos(\theta_{1}) - \phi_{1} \sin(\theta_{1})\} ds_{1} = \int_{0}^{\xi_{2}} \{\cos(\theta_{2}) - \phi_{2} \sin(\theta_{2})\} ds_{2}$$
$$= -\int_{\xi_{1}}^{L_{1}} \{\cos(\theta_{1}) - \phi_{a} \sin(\theta_{1})\} ds_{1}$$
247 (27)

 Solving balance equations (20)–(22) yields six constants of inte- gration,  $c_1, c_2, \ldots, c_6$ . Hence, there are altogether nine unknowns:  $a, \lambda_1, \lambda_2, c_1, c_2, \ldots, c_6 a$ , to be determined by substituting the solu- tions to Eqs. (20)–(22) into Eqs. (23)–(27). Consequently, there is a system of nine equations with nine unknowns.

 Numerical solutions to the system described in Eqs. (20)–(27) are presented in Figs. 3–5. In all three sets of figures, (a) depicts the deformation under a varying compressive load F, (b) the con- tact length a as a function of F, and (c) F as a function of stack height w. The results in (b) and (c) are provided for various values of the adhesion energy  $\gamma$ . Here, the compression distance, or the change in height of the stacked cylinders (equilibrium stack height minus the sum of undeformed cylinders) w, is defined as

$$w = 2(R_1 + R_2) - \int_0^{\xi_1} \sin(\theta_1 + \phi_1) ds_1 + \int_0^{\xi_2} \sin(\theta_2 + \phi_2) ds_2$$
(28)

261

**262** The input parameters  $(k_1, k_2, R_1, R_2, \gamma, F)$  and calculated values **263** (a, w) are all unitless.

#### **264 4** Worked Examples

265 The governing equations are derived using the principle of minimum potential energy. The potential energy functional com-266 prises the strain energy created by elastic bending in both the 267 contacting and noncontacting portions of the cylinders, the poten-268 269 tial energy of the external load F, and the work of adhesion to 270 expose new surfaces. Apart from the standard differential and 271 boundary forms of moment balance (17) and (18), stationarity of **272** the potential energy functional furnishes a jump condition at the **273** edge  $(s_1 = \xi_1, s_2 = \xi_2)$  of contact zone (19). A simpler jump condi-274 tion had previously been derived for adhesion of a single cylinder **275** to a rigid, flat substrate, a result that has recently been shown to be **276** equivalent to a discontinuity in the internal moment [31,35]. The AQ: jump condition in Eq. (19), however, has more terms since it 277 278 concerns adhesion between two generally dissimilar thin-walled 279 cylinders. Moreover, it does not appear to correspond to a discon-**280** tinuity in internal moment and is instead related to a discontinuity 281 in material (configurational) forces or Eshelbian energy-282 momentum.

**283 4.1** Same Stiffness and Radii  $(k_1=k_2, R_1=R_2)$ . Figure 3(*a*) **284** shows the deformed cylinders with  $k_1=k_2=1$  and  $R_1=R_2=1$ , un-**285** der the coupled action of an external compressive load and adhe-

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Fig. 3 Adhesion between two identical cylindrical shells with the same bending stiffness  $(k_1 = k_2 = 1)$  and radii  $(R_1 = R_2 = 1)$  under a compressive load *F* for  $\gamma = 3$  (unless indicated otherwise). (a) Deformed profile with pole of bottom cylinder as reference. (b) Half contact arc length as a function of compressive load. (c) Change in stack height as a function of compressive load *F* and contact length *a*, with the dashed line indicating line contact (*a*=0) where adhesion has no influence on the interacting cylinders.

sion with  $\gamma=3$ . Both cylinders are flattened at their contact inter-286 face and globally deformed to a pseudo-elliptic geometry, with the 287 lower pole of the bottom cylinder as the reference  $(s_1=0)$ . The 288 deformation is symmetric with respect to the planar contact. In 289 this respect, the identical cylinders deform in a manner qualita-290





Fig. 4 Adhesion between two cylindrical shells with the same bending stiffness  $(k_1=k_2=1)$  but different radii  $(R_1=1, R_2=0.5)$  under a compressive load *F* for  $\gamma=3$  (unless indicated otherwise). (a) Deformed profile. (b) Half contact arc length as a function of compressive load. (c) Change in stack height as a function of compressive load and contact length, with the dashed line indicating line contact (a=0) where adhesion has no influence.

Fig. 5 Adhesion between two cylindrical shells with the same radii  $(R_1=R_2=1)$  but different bending stiffnesses  $(k_1=1,k_2=0.5)$  under a compressive load *F* for  $\gamma=3$  (unless indicated otherwise). (*a*) Deformed profile. (*b*) Half arc contact length as a function of compressive load. (*c*) Change in stack height as a function of compressive load and contact length, with the dashed line indicating line contact (a=0) where adhesion has no influence.

 tively similar (but not equivalent) to the adhesion of a single cyl- inder to a flat, rigid substrate. Figures 3(b) and 3(c) show the mechanical responses a(F) and F(w) for a range of  $\gamma$ . As F de- creases, the contact shrinks and continues to be finite even when the external load turns tensile (F < 0). When the tensile load reaches the threshold,  $F_0 = \min(F)$ , the contact vanishes (a=0) and the two adhering cylinders snap, leading to pull-off. The critical 297 tensile load (negative *F*) increases with increasing  $\gamma$ ; e.g.,  $F_0(\gamma 298 = 3) \approx -1.1$  and  $F_0(\gamma=5)=-2$ . Interestingly,  $F_0(\gamma=1)=0$  is pre-299 dicted, implying that the work of adhesion is insufficient to cause 300 spontaneous adhesion of the two cylinders. A minimum compres-301 sive load is necessary to make finite contact (a > 0). In reality, 302

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303 adhesion is the result of intersurface forces with *finite* range such **304** that the cylinders interact even in the absence of intimate contact 305 and a tensile load is always needed to separate the adherends. For **306**  $0 < \gamma < 1$ ,  $F_0$  is positive at a=0 such that adhesion is irrelevant for **307**  $0 < F < F_0$  and  $\Pi$  thus comprises the elastic deformation energy **308** and potential energy due to external load only. Physically, when F 309 falls below  $F_0$ , the contact area remains a line (a=0) until w 310 reduces to zero. There exists a minimal critical cylinder radius  $R_{\min}$ , below which the contact is always zero (to be discussed in 311 Sec. 5). In Fig. 3(c), the compression distance is always positive 312 (w>0) even in the absence of external load (F=0) as adhesion 313 **314** compels the two cylinders. As the load turns tensile (F < 0), w reduces further and the cylinder becomes more elongated about 315 316 the vertical axis until pull-off occurs at the termini of all curves.

**4.2** Same Stiffness and Different Radii  $(k_1=k_2,R_1=2R_2)$ . Figure 4(*a*) shows two dissimilar cylinders with  $k_1=k_2=1$  but  $R_1=1$  and  $R_2=0.5$ . Here, the deformation about the curved contact becomes asymmetric. Elastic deformation is mainly confined to the larger cylinder even along the contact length. The relations a(F) and F(w) are similar to Figs. 3(*b*) and 3(*c*) despite a shift in  $F_0$ . The F(w) for  $\gamma=1$  and 2 terminate as adhesion loses its influ-ence on the cylinders.

325 4.3 Different Stiffness and Same Radii  $(k_1=2k_2,R_1=R_2)$ . **326** Figure 5(*a*) shows dissimilar cylinders with  $k_1 = 1$  and  $k_2 = 0.5$ , but **327**  $R_1 = R_2 = 1$ . The more compliant cylinder suffers from a larger degree of deformation. For F=1, the change in angle  $\phi_2$  is as large 328 as 0.6 rad over much of the noncontacting portion. Hence, the 329 **330** small angle approximation used to derive Eqs. (20)–(22) and (27) 331 is no longer suitable and the exact differential equation or higher 332 order approximation is necessary. Moreover, because of the greater compliance, Figs. 5(b) and 5(c) are limited to  $\gamma \leq 3$ . For 333 **334** larger  $\gamma$ , the more compliant cylinder will spontaneously adhere to 335 the stiffer cylinder and undergo deformation angles  $\phi_2$  that are 336 well beyond the range of the small angle approximation.

337 4.4 Example of Carbon Nanotubes. A practical example is 338 the mechanical deformation of CNT in the presence of adhesion. Though the proper computation should incorporate the crystallo-339 340 graphic structure and orientation, we adopt the present continuum model and compare the results with molecular simulation by Tang 341 342 et al. [22]. A comprehensive summary of published CNT materials parameters is given by Tu and Ou-Yang [25]. According to Sears 343 344 and Batra [32], an equivalent elastic tube representation of CNT 345 possesses sheet thickness h=0.1 nm, radius R=0.6 nm, and elastic modulus E=3.0 TPa. In order to achieve a contact arc length 346 a=0.1 nm (17% of R), an adhesion energy of  $\gamma \approx 1.0$  J m<sup>-2</sup> is 347 required, which is a reasonable estimate of the van der Waals 348 interactions. In the presence of water meniscus alone,  $\gamma$ 349  $\approx 0.144$  J m<sup>-2</sup>, which falls below the critical adhesion energy  $\gamma^*$ , **351** the contact area is a line (a=0), and the corresponding pull-off load vanishes ( $F_0=0$ ). 352

#### 353 5 Discussion

354 It is worthwhile to compare the present model with the classical 355 JKR and DMT models for adhering solid spheres. For example, the predicted pull-off force  $F_0$  is found to have a much stronger 356 dependence on the size (R) than stiffness (k), in reminiscence of 357 358  $(F_0)_{\rm JKR} = (3/2)\pi R\gamma$  and  $(F_0)_{\rm DMT} = \pi R\gamma$ , where both depend only on the solid sphere dimension but not on materials stiffness. To 359 make a more rigorous comparison, the pull-off force is normalized 360 **361** by  $\pi R_{\text{eff}} \gamma$  with  $R_{\text{eff}}^{-1} = R_1^{-1} + R_2^{-1}$  being the effective cylinder ra-**362** dius. Figure 6(*a*) presents  $R_1 = R_2 = 1$  and  $k_1 = 1$  for a range of  $k_2$ . **363** In the limit of large  $\gamma$ ,  $F_0/\pi R \gamma$  approaches an asymptote of approximately 1/4, independent of  $k_1$  and  $k_2$ . On the other hand, 364 **365** once the adhesion energy falls below a threshold of  $\gamma^*$ , pull-off **366** occurs at  $F_0(\gamma \le \gamma^*) = 0$ . Despite the similarity with JKR, cylindri-

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Fig. 6 Normalized pull-off strength  $F_0$  as a function of adhesion energy  $\gamma$ . (a) Same radius ( $R_1=R_2=1$ ), but different stiffness with  $k_1=1$ . (b) Same stiffness ( $k_1=k_2=1$ ), but different radii with  $R_1=1$ .

cal shells deviate significantly at small  $\gamma$ . Figure 6(b) shows  $k_1$  367 = $k_2$ =1 and  $R_1$ =1 for a range of  $R_2$ . The monotonic increasing  $F_0$  368 again shows a minimum threshold with  $F_0(\gamma \leq \gamma^*)=0$ , but there 369 does not exist a common threshold for large  $\gamma$ . The fact that the 370 upper limit for each  $F_0(\gamma)$  curve decreases with increasing  $R_2$  371 indicates that (i) the larger cylinder becomes more compliant and 372 thus requires a smaller pull-off force, and (ii) the pull-off depends 373 predominantly on the cylinder dimension alluding to the JKR 374 model. 375

A fundamental difference between the current model and JKR 376 is noted. A distinct feature of the JKR is the local deformation of 377 the adhering spheres at the contact circle. In essence, the com- 378 bined applied load and adhesion force press a sphere against a 379 rigid planar substrate to create a Hertz contact circle. While re- 380 taining the contact circle, the adhesion force is then removed and 381 replaced with a local deformation around the contact circle. This 382 is done by assuming that a circular punch in full contact with a 383 half elastic continuum pulls on the substrate giving rise to a linear 384 "relaxation" and reduction in the approach displacement. Proper 385 energy balance thus leads to a mechanical instability or pull-off at 386 a critical tensile applied load and a nonzero contact radius. Should 387 the essential relaxation be ignored, the contact circle always 388 shrinks to zero (one-point contact) at pull-off. In general, the char- 389 acteristic nonzero pull-off contact is expected in geometrically 390 incompatible surfaces (e.g., spheres). Existing models for thin- 391 walled vesicles ignore local deformation and indeed predict zero 392 pull-off radius [33–35], though it must be emphasized that deter- 393 mination of the exact pull-off radius proves to be quite elusive. 394 Nevertheless, our present model does not consider local deforma- 395



Fig. 7 Threshold radius for line contact (a=0) as function of stiffness and adhesion energy.  $(R_{\min}=(k/\gamma)^{1/2} \text{ for } R_1=R_2=R \text{ and}$  $k_1 = k_2 = k$ .)

396 tion and the contact arc length must therefore reduce to zero (i.e., 397 line contact) at pull-off. A comprehensive model is beyond the 398 scope of this paper.

399 The eccentric behavior of line contact (a=0) present at small 400 compressive external load (cf. Fig. 3(c)) is worth discussing. 401 Based on the assumption that carbon nanotubes are planar graphene sheets folded into a cylindrical shell, Hui and co-402 403 workers [22,23] used an alternative method to derive a minimum 404 cylindrical radius below which the adhesion contact remains a **405** line:  $R_{\min} = (k/\gamma)^{1/2}$  for  $R_1 = R_2 = R$  and  $k_1 = k_2 = k$ . Compliant cyl-**406** inders (small k) coupled with strong adhesion (large  $\gamma$ ) is more prone to deformation and thus a small  $R_{\min}$ . The present model 407 **408** considers cylinders are initially stress-free. To deduce the relation between  $R_{\min}$ , k, and  $\gamma$ , values of R and k are randomly chosen, 409 **410** and the relation a(F) is then found for a range of  $\gamma$ . The unique **411** curve intersecting the origin (a=0 and F=0) corresponds to the **412** value of  $\gamma(R_{\min})$ . For instance, in Fig. 3(b), k=1 and  $R_{\min}=1$ ; **413** therefore,  $\gamma = 1$  because the corresponding a(F) intersects the ori-414 gin. The numerical routine is repeated for a range of k and Rcombinations. Notwithstanding the distinctly different assump-415 416 tions and analyses in the two models, an excellent comparison 417 between our present model (data) and that of Hui and co-workers **418** (curves) is shown in Fig. 7 for  $R_{\min}$  as a function of k for specific **419**  $\gamma$ . The consistency is expected because no matter the cylinders 420 possess an intrinsic stress, mechanical deformation to form the 421 planar contact area causes a compressive stress to build up within **422** the contact and immediately without, and thus raise the elastic 423 energy of the system from the ground state of undeformed geom-424 etry. The current model is more general in the sense that dissimilar **425** cylinders with different stiffnesses and dimensions are considered. 426 Moreover, we deduce that the nonzero  $R_{\min}$  is a consequence of 427 the global deformation of the cylinders and the local deformation within the contact arc length, instead of "a residual stress that 428 429 increases the stiffness of smaller diameter tubes" [22]. 430 Our present 2D cylindrical shell model sheds lights on the ad-431 hesion of 3D structures. One application is in cell aggregation, **432** which is related to the formation and growth of natural, prosthetic,

**433** and malignant tissues. The existing model in literature treats cells 434 as deformable solid spheres conforming to JKR theory. When 435 these free entities come into contact due to thermal collision and 436 vibration, interfacial adhesion occurs, followed by aggregation 437 and coagulation. It is again emphasized that cells are not solid 438 spheres but a viscoelastic cytoplasm encapsulated by a thin lipid **439** bilayer membrane (shell). The present adhesion model properly 440 addresses the nature of coupled shell deformation and adhesion, **441** provides the constitutive relations between *F*, *w*, and *a*, and thus yields the basis for a correct statistical portrayal of the Gibbs free 442 energy and partition function of the grand canonical ensemble 443 [36]. Immediate biomedical application is found in deriving the 444 physical thresholds for cell aggregation (e.g., concentration, di- 445 mension, and temperature). Long-range surface forces can also be 446 incorporated into the present model such that the adhering sur- 447 faces sense the presence of their counterpart even prior to direct 448 contact, as shown in our latest work for freestanding membrane 449 clamped at the periphery adhering to a planar substrate [20]. 450

#### Conclusion 6

Using an energy balance, we derived the adhesion mechanics 452 for two interacting elastic cylindrical shells with ranges of bend- 453 ing stiffness, radii, and adhesion energy. Relationships are estab- 454 lished between the measurable quantities at equilibrium, namely, 455 applied load, stack height, contact length and deformed cylinder 456 profiles, and the quasistatic adhesion-delamination trajectories. 457 The graphs and trends presented have significant implications in 458 the adhesion of similar and dissimilar interfaces in micro-/ 459 nanoshell structures. Such interactions are relevant to a variety of 460 systems in nanoscience and technology, life-sciences, and tissue 461 engineering. 462

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463

451

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#### Appendix A

468

Following the definitions in Eqs. (14) and (16), the total poten- 469 tial energy of the system may be expressed as 470

$$\Pi = \int_{0}^{\xi_{1}} \Lambda_{1} ds_{1} + \int_{0}^{\xi_{2}} \Lambda_{2} ds_{2} + \int_{\xi_{1}}^{L_{1}} \Lambda_{a} ds_{1}$$
(A1) **471**

At equilibrium,  $\Pi$  must be stationary with respect to variations in 472  $\phi_1, \phi_2$ , and  $\phi_a$  as well as their derivatives ( $\phi_{1,1}, \phi_{2,2}$ , and  $\phi_{a,1}$ ). 473 Applying these variations simultaneously to  $\Pi$  yields an expres- 474 sion of the form 475

$$\delta \Pi_{\phi} = \int_{0}^{\xi_{1}} \left\{ \frac{\partial \Lambda_{1}}{\partial \phi_{1}} \delta \phi_{1} + \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \delta \phi_{1,1} \right\} ds_{1} + \int_{0}^{\xi_{2}} \left\{ \frac{\partial \Lambda_{2}}{\partial \phi_{2}} \delta \phi_{2} + \frac{\partial \Lambda_{2}}{\partial \phi_{2,2}} \delta \phi_{2,2} \right\} ds_{2} + \int_{\xi_{1}}^{L_{1}} \left\{ \frac{\partial \Lambda_{a}}{\partial \phi_{a}} \delta \phi_{a} + \frac{\partial \Lambda_{a}}{\partial \phi_{a,1}} \delta \phi_{a,1} \right\} ds_{1}$$

$$(\Lambda^{2}) \Lambda^{77}$$

By the chain rule,

478

$$\int_{0}^{\xi_{1}} \left\{ \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \delta \phi_{1,1} \right\} ds_{1}$$

$$= \int_{0}^{\xi_{1}} \left\{ \frac{d}{ds} \left( \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \delta \phi_{1} \right) - \frac{d}{ds} \left( \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \right) \delta \phi_{1} \right\} ds_{1}$$
480

$$= \left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}} \delta \phi_1\right)_{s_1 = \xi_1} - \left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}} \delta \phi_1\right)_{s_1 = 0}$$

$$- \int_0^{\xi_1} \left\{\frac{d}{ds} \left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}}\right) \delta \phi_1\right\} ds_1$$
(A3)
(A3)

$$\int \left\{ \frac{1}{ds} \left( \frac{1}{\partial \phi_{1,1}} \right) \delta \phi_1 \right\} ds_1$$
(A3)
482

Here, the operation  $(f)_{x=y}$  denotes the value of f at x=y. Applying 483 this same identity to the other integrals in Eq. (A2), 484

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$$\delta \Pi_{\phi} = \left(\frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \delta \phi_{1}\right)_{s_{1}=\xi_{1}} - \left(\frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \delta \phi_{1}\right)_{s_{1}=0} + \int_{0}^{\xi_{1}} \left\{\frac{\partial \Lambda_{1}}{\partial \phi_{1}} - \frac{d}{ds} \left(\frac{\partial \Lambda_{1}}{\partial \phi_{1,1}}\right)\right\} \delta \phi_{1} ds_{1} \left(\frac{\partial \Lambda_{2}}{\partial \phi_{2,2}} \delta \phi_{2}\right)_{s_{2}=\xi_{2}} - \left(\frac{\partial \Lambda_{2}}{\partial \phi_{2,2}} \delta \phi_{2}\right)_{s_{2}=0} + \int_{0}^{\xi_{2}} \left\{\frac{\partial \Lambda_{2}}{\partial \phi_{2}} - \frac{d}{ds} \left(\frac{\partial \Lambda_{2}}{\partial \phi_{2,2}}\right)\right\} \delta \phi_{2} ds_{2} \left(\frac{\partial \Lambda_{a}}{\partial \phi_{a,1}} \delta \phi_{a}\right)_{s_{1}=L_{1}} - \left(\frac{\partial \Lambda_{a}}{\partial \phi_{a,1}} \delta \phi_{a}\right)_{s_{1}=\xi_{1}} + \int_{\xi_{1}}^{L_{1}} \left\{\frac{\partial \Lambda_{a}}{\partial \phi_{a}} - \frac{d}{ds} \left(\frac{\partial \Lambda_{a}}{\partial \phi_{a,1}}\right)\right\} \delta \phi_{a} ds_{1}$$
  
488 (A4)

**489** At equilibrium,  $\delta \Pi_{\phi}$  must vanish for *kinematically admissible* **490** variations in  $\phi_1$ ,  $\phi_2$ , and  $\phi_a$ .

**491** According to the boundary conditions in Eq. (7), both  $\phi_1(0)$ **492** =  $\phi_2(0) = \phi_a(L_1) = 0$  and

$$\phi_1(0) + \delta\phi_1(0) = \phi_2(0) + \delta\phi_2(0) = \phi_a(L_1) + \delta\phi_a(L_1) = 0$$
(A5)

 must be satisfied. Clearly, this implies  $\delta \phi_1(0) = \delta \phi_2(0) = \delta \phi_a(L_1)$ ; in other words, variations in the deflection  $\phi_i$  must vanish at the points  $s_i$  where  $\phi_i$  is prescribed. Similarly, the boundary condi-tions in Eq. (8) require that both the conditions

**498** 
$$\phi_a(\xi_1) = \phi_1(\xi_1) = 2\pi + \phi_2(\xi_2) + \theta_2(\xi_2) - \theta_1(\xi_1)$$
 (A6)

499 and

501

493

500  $\phi_a(\xi_1) + \delta \phi_a(\xi_1) = \phi_1(\xi_1) + \delta \phi_1(\xi_1)$  $= 2\pi + \phi_2(\xi_2) + \delta \phi_2(\xi_2) + \theta_2(\xi_2) - \theta_1(\xi_1)$ 

**502** be satisfied. This implies  $\delta \phi_a(\xi_1) = \delta \phi_1(\xi_1) = \delta \phi_2(\xi_2) = \delta \phi_{\xi}$ . Sub-**503** stituting the boundary conditions expressions for  $\delta \phi_i$  into Eq. **504** (A4),

$$\delta \Pi_{\phi} = \left\{ \left( \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \right)_{s_{1} = \xi_{1}} + \left( \frac{\partial \Lambda_{2}}{\partial \phi_{2,2}} \right)_{s_{2} = \xi_{2}} - \left( \frac{\partial \Lambda_{a}}{\partial \phi_{a,1}} \right)_{s_{1} = \xi_{1}} \right\} \delta \phi_{\xi}$$

$$+ \int_{0}^{\xi_{1}} \left\{ \frac{\partial \Lambda_{1}}{\partial \phi_{1}} - \frac{d}{ds} \left( \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \right) \right\} \delta \phi_{1} ds_{1} + \int_{0}^{\xi_{2}} \left\{ \frac{\partial \Lambda_{2}}{\partial \phi_{2}} - \frac{d}{d\phi_{2}} \right\} ds_{1} ds_{1}$$

507

$$-\frac{d}{ds}\left(\frac{\partial\Lambda_2}{\partial\phi_{2,2}}\right)\bigg\}\delta\phi_2ds_2 + \int_{\xi_1}^{1}\bigg\{\frac{\partial\Lambda_a}{\partial\phi_a} - \frac{d}{ds}\bigg(\frac{\partial\Lambda_a}{\partial\phi_{a,1}}\bigg)\bigg\}\delta\phi_ads_1$$
(A8)

**508** At this point, the variations  $\delta \phi_{\xi}$ ,  $\delta \phi_1$ ,  $\delta \phi_2$ , and  $\delta \phi_a$  are all inde-**509** pendent and arbitrary. Hence, in order for  $\delta \Pi_{\phi}$  to vanish, the **510** conditions

**511** 
$$\frac{\partial \Lambda_1}{\partial \phi_1} - \frac{d}{ds_1} \left( \frac{\partial \Lambda_1}{\partial \phi_{1,1}} \right) = 0, \quad \frac{\partial \Lambda_2}{\partial \phi_2} - \frac{d}{ds_2} \left( \frac{\partial \Lambda_2}{\partial \phi_{2,2}} \right) = 0,$$

512

513

$$\left(\frac{\partial\Lambda_1}{\partial\phi_{1,1}}\right)_{s_1=\xi_1} + \left(\frac{\partial\Lambda_2}{\partial\phi_{2,2}}\right)_{s_2=\xi_2} - \left(\frac{\partial\Lambda_a}{\partial\phi_{a,1}}\right)_{s_1=\xi_1} = 0 \qquad (A10)$$

 $\frac{\partial \Lambda_a}{\partial \phi_a} - \frac{d}{ds_1} \left( \frac{\partial \Lambda_a}{\partial \phi_{a,1}} \right) = 0$ 

**514** must be satisfied. It is important to note that boundary condition **515** (A10) results from simultaneously applying the first three varia-**516** tions in Eq. (15). This is necessary since the variations are not **517** independent, but related through the boundary conditions (7) and **518** (8). Failure to incorporate these conditions into the calculus of **519** variations would either eliminate kinematic constraints or intro-**520** duce nonexistent ones.

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### Appendix B

At equilibrium,  $\Pi$  must be stationary with respect to variations 522 in *a*. Since  $\delta a$  is arbitrary, the corresponding variation in  $\Pi$ , 523  $\delta \Pi_a = (d\Pi/da) \delta a$ , vanishes if and only if  $d\Pi/da = 0$ . Since both  $\xi_1$  524 and  $\xi_2$  depend on *a*,  $d\Pi/da$  must be evaluated using Leibniz' 525 integration rule 526

$$\frac{d\Pi}{da} = (\Lambda_1)_{s_1 = \xi_1} \frac{d\xi_1}{da} + (\Lambda_2)_{s_2 = \xi_2} \frac{d\xi_2}{da} - (\Lambda_a)_{s_1 = \xi_1} \frac{d\xi_1}{da} + \int_0^{\xi_1} \left\{ \frac{\partial \Lambda_1}{\partial \phi_1} \frac{d\phi_1}{da} \right\}_{s_1 = \xi_1} \frac{d\xi_1}{da} + \int_0^{\xi_2} \left\{ \frac{\partial \Lambda_2}{\partial \phi_2} \frac{d\phi_2}{da} + \frac{\partial \Lambda_2}{\partial \phi_{2,2}} \frac{d\phi_{2,2}}{da} \right\}_{s_2} \frac{d\xi_2}{da} + \int_{\xi_1}^{L_1} \left\{ \frac{\partial \Lambda_a}{\partial \phi_a} \frac{d\phi_a}{da} + \frac{\partial \Lambda_a}{\partial \phi_{a,1}} \frac{d\phi_{a,1}}{da} \right\}_{s_1}$$
(B1)

In light of the balance law (17),

(A7)

(A9)

$$\int_{0}^{\xi_{1}} \left\{ \frac{\partial \Lambda_{1}}{\partial \phi_{1}} \frac{d\phi_{1}}{da} + \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \frac{d\phi_{1,1}}{da} \right\} ds_{1}$$
531

$$= \int_{0}^{\xi_{1}} \left\{ \frac{d}{ds_{1}} \left( \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \right) \frac{d\phi_{1}}{da} + \frac{\partial \Lambda_{1}}{\partial \phi_{1,1}} \frac{d\phi_{1,1}}{da} \right\} ds_{1}$$
532

$$= \int_{0}^{\xi_1} \frac{d}{ds_1} \left\{ \frac{\partial \Lambda_1}{\partial \phi_{1,1}} \frac{d\phi_1}{da} \right\} ds_1$$
533

$$= \left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}} \frac{d\phi_1}{da}\right)_{s_1 = \xi_1} - \left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}} \frac{d\phi_1}{da}\right)_{s_1 = 0}$$
(B2) 534

Applying this identity to the other two integrals in Eq. (B1) and 535 noting that  $d\xi_1/da=d\xi_2/da=-1$ , 536

$$\frac{d\Pi}{da} = (\Lambda_a)_{s_1 = \xi_1} - (\Lambda_1)_{s_1 = \xi_1} - (\Lambda_2)_{s_2 = \xi_2} + \left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}} \frac{d \phi_1}{da}\right)_{s_1 = \xi_1}$$
537

$$\left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}} \frac{d\phi_1}{da}\right)_{s_1=0} + \left(\frac{\partial \Lambda_2}{\partial \phi_{2,2}} \frac{d\phi_2}{da}\right)_{s_2=\xi_2} - \left(\frac{\partial \Lambda_2}{\partial \phi_{2,2}} \frac{d\phi_2}{da}\right)_{s_2=0}$$
538

$$+ \left(\frac{\partial \Lambda_a}{\partial \phi_{a,1}} \frac{d\phi_a}{da}\right)_{s_1 = L} - \left(\frac{\partial \Lambda_a}{\partial \phi_{a,1}} \frac{d\phi_a}{da}\right)_{s_1 = \xi_1}$$
(B3)

Next, by natural boundary condition (18),  $d\Pi/da$  reduces to 540

$$\frac{d\Pi}{da} = (\Lambda_a)_{s_1 = \xi_1} - (\Lambda_1)_{s_1 = \xi_1} - (\Lambda_2)_{s_2 = \xi_2}$$
541

$$+\left\{\frac{\partial\Lambda_1}{\partial\phi_{1,1}}\left(\frac{d\phi_1}{da}-\frac{d\phi_a}{da}\right)\right\}_{s_1=\xi_1}-\left(\frac{\partial\Lambda_1}{\partial\phi_{1,1}}\frac{d\phi_1}{da}\right)_{s_1=0}$$
542

$$+ \left(\frac{\partial \Lambda_2}{\partial \phi_{2,2}}\right)_{s_2 = \xi_2} \left\{ \left(\frac{d\phi_2}{da}\right)_{s_2 = \xi_2} - \left(\frac{d\phi_a}{da}\right)_{s_1 = \xi_1} \right\}$$
543

$$-\left(\frac{\partial\Lambda_2}{\partial\phi_{2,2}}\frac{d\phi_2}{da}\right)_{s_2=0} + \left(\frac{\partial\Lambda_a}{\partial\phi_{a,1}}\frac{d\phi_a}{da}\right)_{s_1=L_1}$$
(B4) 54

Equation (B4) represents the general expression for  $d\Pi/da$  at 545 equilibrium. It can be further simplified by applying the boundary 546 conditions in Eqs. (7) and (8), which restrict not only  $\phi_1$ ,  $\phi_2$ , and 547  $\phi_a$ , but also their derivatives with respect to *a*. According to Eq. 548 (7),  $\phi_1$ ,  $\phi_2$ , and  $\phi_a$  are all prescribed at  $s_1=0$ ,  $s_2=0$ , and  $s_1=L_1$ , 549 respectively. Since these conditions must hold for all values of *a*, 550 the derivatives  $(d\phi_1/da)_{s_1=0}$ ,  $(d\phi_2/da)_{s_2=0}$ , and  $(d\phi_a/da)_{s_1=L_1}$  551 must equal zero. This is equivalent to the condition in the calculus 552 of variations that the variation of prescribed end points must van-554

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In contrast,  $(d\phi_1/da)_{s_1=\xi_1}$ ,  $(d\phi_2/da)_{s_2=\xi_2}$ , and  $(d\phi_a/da)_{s_1=\xi_1}$  are 555 556 nonzero and must be computed using the boundary conditions in 557 Eq. (8). According to Eq. (15) and the fundamental theorem of 558 calculus,

559 
$$\phi_1(\xi_1) = \phi_1^*(\xi_1) + \delta\phi_1(\xi_1) = \phi_1^*(\xi_1^*) - \delta a \phi_{1,1}^*(\xi_1^*) + \delta\phi_1(\xi_1)$$
  
560  $+ O(\delta a^2)$  (B5)

**561** where  $\xi_1^* = L_1 - a^*$  and  $a^*$  is the value of a at equilibrium. Simi-**562** larly,

$$\phi_a(\xi_1) = \phi_a^{*}(\xi_1^{*}) - \delta a \phi_{a,1}^{*}(\xi_1^{*}) + \delta \phi_a(\xi_1) + O(\delta a^2)$$
 (B6)

564 which, according to boundary condition (8), must be equivalent to 565  $\phi_1(\xi_1)$ . Since  $\delta a$  is infinitesimally small, terms of order  $O(\delta a^2)$ **566** may be omitted and so the conditions  $\phi_1(\xi_1) = \phi_a(\xi_1)$  and 567  $\phi_1(\xi_1^*) = \phi_a(\xi_1^*)$  together imply

568 
$$-\delta a \phi_{1,1}^{*}(\xi_{1}^{*}) + \delta \phi_{1}(\xi_{1}) = -\delta a \phi_{a,1}^{*}(\xi_{1}^{*}) + \delta \phi_{a}(\xi_{1})$$
(B7)

**569** Dividing both sides by  $\delta a$ , taking the limit as  $\delta a \rightarrow 0$ , and then **570** rearranging terms,

$$\left(\frac{d\phi_a}{da}\right)_{s_1=\xi_1} - \left(\frac{d\phi_1}{da}\right)_{s_1=\xi_1} = \phi_{a,1}(\xi_1) - \phi_{1,1}(\xi_1)$$
(B8)

572 where the asterisk denoting the value at equilibrium is henceforth **573** omitted. Using the same argument for  $\phi_2(\xi_2)$ , it follows from Eq. **574** (8) that

$$\left(\frac{d\phi_a}{da}\right)_{s_1=\xi_1} - \left(\frac{d\phi_2}{da}\right)_{s_2=\xi_2} = \phi_{a,1}(\xi_1) - \phi_{2,2}(\xi_2) + \frac{1}{R_1} + \frac{1}{R_2}$$
(B9)

575

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563

**576** Substituting this into Eq. (19) and setting  $d\Pi/da$  equal to zero **577** yield the following jump condition at equilibrium:

578 
$$(\Lambda_a)_{s_1 = \xi_1} - (\Lambda_1)_{s_1 = \xi_1} - (\Lambda_2)_{s_2 = \xi_2} + \left(\frac{\partial \Lambda_1}{\partial \phi_{1,1}}\right)_{s_1 = \xi_1} \{\phi_{1,1}(\xi_1) - \phi_{a,1}(\xi_1)\}$$

579 
$$+ \left(\frac{\partial \Lambda_2}{\partial \phi_{2,2}}\right)_{s_2 = \xi_2} \left\{ \phi_{2,2}(\xi_2) - \phi_{a,1}(\xi_1) - \frac{1}{R_1} - \frac{1}{R_2} \right\} = 0 \qquad (B10)$$

#### 580 Appendix C

581 Assuming that the deflections  $\phi_1$ ,  $\phi_2$ , and  $\phi_a$  are small, the 582 Lagrangian densities may be approximated as

**583** 
$$\Lambda_1 = \frac{1}{2}k_1\phi_{1,1}^2 + F\{\sin(\theta_1) + \phi_1\cos(\theta_1)\} + \lambda_1\{\cos(\theta_1) - \phi_1\sin(\theta_1)\}$$

584 
$$\Lambda_2 = \frac{1}{2}k_2\phi_{2,2}^2 - F\{\sin(\theta_2) + \phi_2\cos(\theta_2)\} + \lambda_2\{\cos(\theta_2) - \phi_2\sin(\theta_2)\}$$

585 
$$\Lambda_a = \frac{1}{2}(k_1 + k_2)\phi_{a,1}^2 + k_2\phi_{a,1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{1}{2}k_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^2$$

586 + 
$$(\lambda_1 + \lambda_2) \{\cos(\theta_1) - \phi_a \sin(\theta_1)\} - \gamma$$
 (C1)

587 The balance equations are obtained by substituting these expres-**588** sions into the Euler–Lagrange differential equation (17). This 589 yields

**590** 
$$k_1 \phi_{1,11} = F \cos(\theta_1) - \lambda_1 \sin(\theta_1)$$
 (C2)

**591** 
$$k_2 \phi_{2,22} = -F \cos(\theta_2) - \lambda_2 \sin(\theta_2)$$
 (C3)

(k<sub>1</sub> + k<sub>2</sub>)
$$\phi_{a,11} = -(\lambda_1 + \lambda_2)\sin(\theta_1)$$
 (C4)

**593** where  $\phi_{i,jj} = d^2 \phi_i / ds_j^2$ .

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