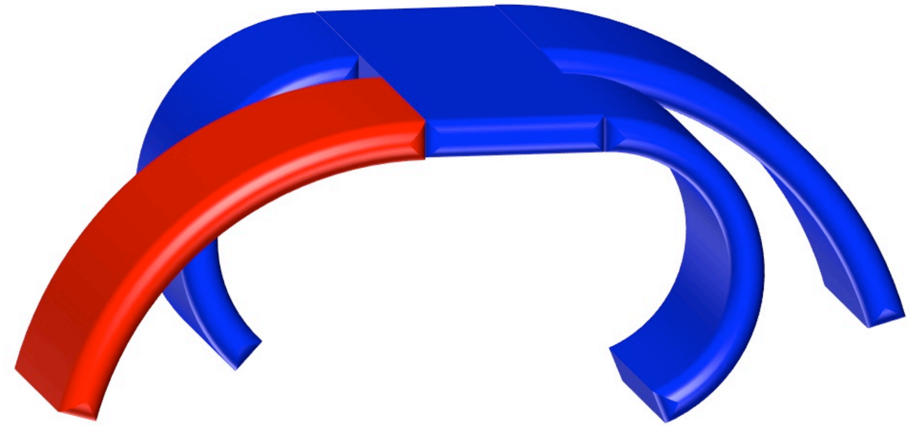
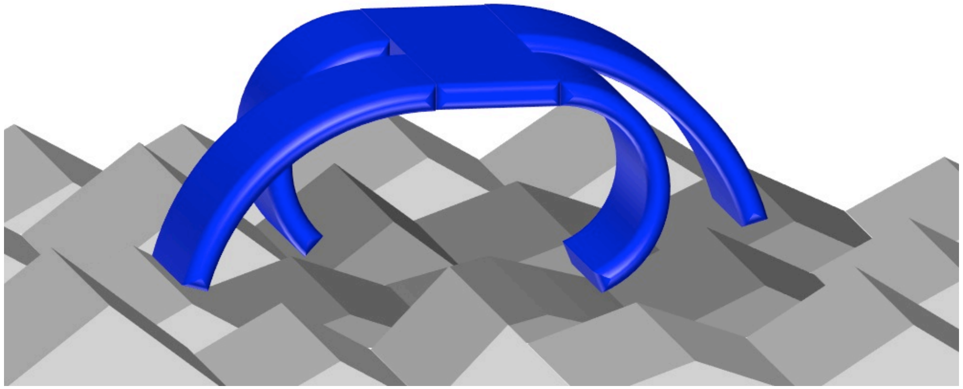
SOFT ROBOT SIMULATION



24-673: Soft Matter Engineering
for Physical Human-Machine Interaction

Prof. Carmel Majidi

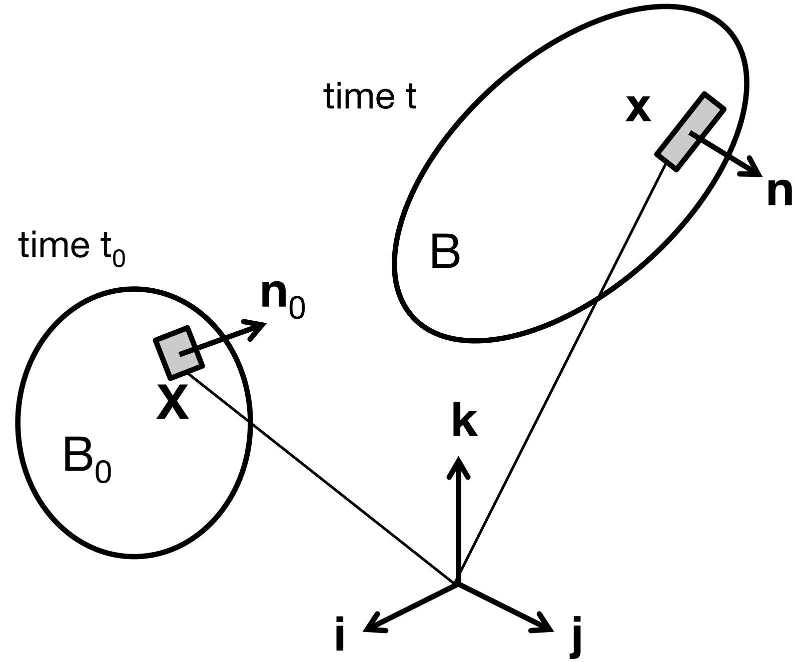


Soft Robot Simulation

- **Composition of Robot** – Materials & constitutive properties
    - Elastomers, fluids, gas, rigid elements
    - Hyperelasticity – coefficients of elasticity, Poisson's ratio
    - Dielectric, ferroelectric, or piezoelectric properties (e.g. electric permittivity)
    - Ferromagnetic properties
    - Shape memory or thermal properties (e.g. coefficient of thermal expansion)
- **State of Robot** – Material shape and condition
    - Kinematics – shape & velocities
        - *Reference placement* – initial shape at time t_0; composed of points $\mathbf{X} \in B_0$
        - *Current placement* – current shape at time t; composed of points $\mathbf{x} \in B$
        - Displacement: $\mathbf{u} = \mathbf{x} - \mathbf{X}$
    - Internal voltage field, temperature distribution, magnetic state, …

- **Physical Interactions** – External loads and environmental conditions
  - Mechanical – contact forces (unilateral constraints, friction, collisions), fluid pressure, gravity
  - Electrical – applied electrical field or current (e.g. Maxwell stress, magnetic force)
  - Thermal – temperature change, supplied heat
- **Governing Physics** – Balance Laws
  - *Thermodynamics* – 1st & 2nd Laws; Principle of Minimum Potential
  - *Newton-Euler Equations* – linear and angular momentum balance for entire robot as well as each volumetric or surface element
  - *Maxwell Equations* – balance of electric displacement and magnetic field

- **Position/Orientation of Robot** – Global Coordinate Systems (COOS)
  - *Lagrangian Description* – coordinates of initial shape (t_0): $\mathbf{X} = X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k}$
    $\nabla_L = (\partial/\partial X_1)\mathbf{i} + (\partial/\partial X_2)\mathbf{j} + (\partial/\partial X_3)\mathbf{k}$
  - *Eulerian Description* – coordinates of current shape (t): $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$
    $\nabla_E = (\partial/\partial x_1)\mathbf{i} + (\partial/\partial x_2)\mathbf{j} + (\partial/\partial x_3)\mathbf{k}$



"Deformation gradient": $\mathbf{F} = \nabla_L\mathbf{x}$

- used to calculate the strain energy density: $W = W(\mathbf{F})$
- Relates the gradient operators: $\nabla_L = \nabla_E\mathbf{F}$
- Relates final and initial volumes: $dV = JdV_0$, where $J = \det(\mathbf{F})$
- $T^{1PK} = \partial W/\partial\mathbf{F}$ is the "1st Piola Kirchoff Stress tensor"
- By definition, Cauchy stress $\sigma = J^{-1}\mathbf{T}^{1PK}\mathbf{F}^T$

The deformation gradient also relates the deformation of surface elements and divergence of the stress tensors:

Consider a surface element that deforms from d$\mathbf{A}_0$ = $\mathbf{n}_0$dA_0 to d$\mathbf{A}$ = $\mathbf{n}$dA. According to Nanson's formula, d$\mathbf{A}$ = J$\mathbf{F}^{-T}$d$\mathbf{A}_0$. This implies the following identity:

$$\underline{T}^{1PK} \cdot d\underline{A}_0 = \underline{\sigma} \cdot d\underline{A}$$

Another useful identity is the Piola transformation (related to Piola identity):

$$\left(\nabla_L \cdot \underline{T}^{1PK}\right) dV_0 = \left(\nabla_E \cdot \underline{\sigma}\right) dV$$

ref. A. Betram, Elasticity & Plasticity of Large Deformations 2[nd] Ed. (2008)

Example: DEA System

$$\Pi = \int_{B_0} W dV_0 - \int_{B} \left\{ \frac{1}{2} \underline{D} \cdot \underline{E} \right\} dV + \int_{\partial B} \left\{ \eta \Phi - \underline{t} \cdot \underline{x} \right\} dA$$

W = strain energy density
$\Phi = \Phi(x_1, x_2, x_3)$ = voltage
$\underline{E} = \nabla_E \Phi$ = electric field
$\underline{\varepsilon}$ = electric permittivity (2nd order tensor; could be anisotropic)
$\underline{D} = \underline{\varepsilon} \cdot \underline{E}$ = electric displacement
η = surface charge (i.e. charge q per unit area)
$\underline{t}$ = surface traction (stress applied to surface)

- $\Pi = \Pi(\underline{x}, \Phi)$ – find position and voltage field that minimizes potential energy.
- Determine change in Π when $\underline{x} \rightarrow \underline{x} + \delta\underline{x}$ and $\Phi \rightarrow \Phi + \delta\Phi$.
- At equilibrium, corresponding change $\delta\Pi = 0$.

Note that $\underline{x} \rightarrow \underline{x} + \delta\underline{x}$ and $\Phi \rightarrow \Phi + \delta\Phi$ imply that
$\nabla_L \underline{x} \rightarrow \nabla_L \underline{x} + \nabla_L \delta\underline{x}$ and $\nabla_E \Phi \rightarrow \nabla_E \Phi + \nabla_E \delta\Phi$

$$\delta\Pi = \int_{B_0} \left\{ T^{1PK} : \left(\delta\nabla_L \underline{x} \right) \right\} dV_0 - \int_B \left\{ \frac{1}{2} \underline{D} \cdot \left(\delta\nabla_E \Phi \right) \right\} dV + \int_{\partial B} \left\{ \eta\delta\Phi - \underline{t} \cdot \delta\underline{x} \right\} dA$$

Product Rule:

$$\int_{B_0} \nabla_L \left\{ T^{1PK} \delta\underline{x} \right\} dV_0 = \int_{B_0} \left\{ \left(\nabla_L \cdot T^{1PK} \right) \delta\underline{x} \right\} dV_0 + \int_{B_0} \left\{ T^{1PK} : \left(\nabla_L \delta\underline{x} \right) \right\} dV_0$$

$$\int_B \nabla_E \left\{ \underline{D} \delta\Phi \right\} dV = \int_B \left\{ \left(\nabla_E \cdot \underline{D} \right) \delta\Phi \right\} dV + \int_B \left\{ \underline{D} \cdot \left(\nabla_E \delta\Phi \right) \right\} dV$$

Divergence Rule: $$\int_{B_0} \nabla_L \left\{ \underline{T}^{1PK} \delta \underline{x} \right\} dV_0 = \int_{\partial B_0} \left\{ \left(\underline{T}^{1PK} \cdot \underline{n}_0 \right) \delta \underline{x} \right\} dA_0$$

$$\int_{B} \nabla_E \left\{ \underline{D} \delta \Phi \right\} dV = \int_{\partial B} \left\{ \left(\underline{D} \cdot \underline{n} \right) \delta \Phi \right\} dA$$

$$\Rightarrow \int_{B} \left\{ \underline{D} \cdot \left(\nabla \delta \Phi \right) \right\} dV = \int_{\partial B} \left\{ \left(\underline{D} \cdot \underline{n} \right) \delta \Phi \right\} dA - \int_{B} \left\{ \left(\nabla \cdot \underline{D} \right) \delta \Phi \right\} dV$$

$$\Rightarrow \int_{B_0} \left\{ T^{1PK} : \left(\nabla_L \delta \underline{x} \right) \right\} dV_0 = \int_{\partial B_0} \left\{ \left(\underline{T}^{1PK} \cdot \underline{n}_0 \right) \delta \underline{x} \right\} dA_0 - \int_{B_0} \left\{ \left(\nabla_L \cdot T^{1PK} \right) \delta \underline{x} \right\} dV_0$$

$$= \int_{\partial B} \left\{ \left(\underline{\sigma} \cdot \underline{n} \right) \delta \underline{x} \right\} dA - \int_{B} \left\{ \left(\nabla_E \cdot \underline{\sigma} \right) \delta \underline{x} \right\} dV$$

Nanson & Piola identities

Substitute these back into the expression for $\delta\Pi$:

$$\delta\Pi = -\int_{B} \left\{ \left(\nabla_{E} \cdot \underline{\sigma} \right) \delta \underline{x} + \left(\nabla_{E} \cdot \underline{D} \right) \delta \Phi \right\} dV$$

$$+ \int_{\partial B} \left\{ \left(\underline{D} \cdot \underline{n} + \eta \right) \delta \Phi + \left(\underline{\sigma} \cdot \underline{n} - \underline{t} \right) \cdot \delta \underline{x} \right\} dA$$

At equilibrium, $\delta\Pi$ must vanish for any arbitrary choice of $\delta\underline{x}$ and $\delta\Phi$.
This implies the following Balance Laws:

$\nabla_E \cdot \underline{\sigma} = 0$ for $\mathbf{x} \in B$ $\quad$ $\underline{\sigma} \cdot \underline{n} = \underline{t}$ for $\mathbf{x} \in \partial B$

$\nabla_E \cdot \underline{D} = 0$ for $\mathbf{x} \in B$ $\quad$ $\underline{D} \cdot \underline{n} = -\eta$ for $\mathbf{x} \in \partial B$

Therefore, finding the displacements and voltages becomes a matter of solving a system of PDEs (“Strong Form”).

The PDEs are either satisfied or not – we can’t evaluate the accuracy of an approximate solution with a single number.

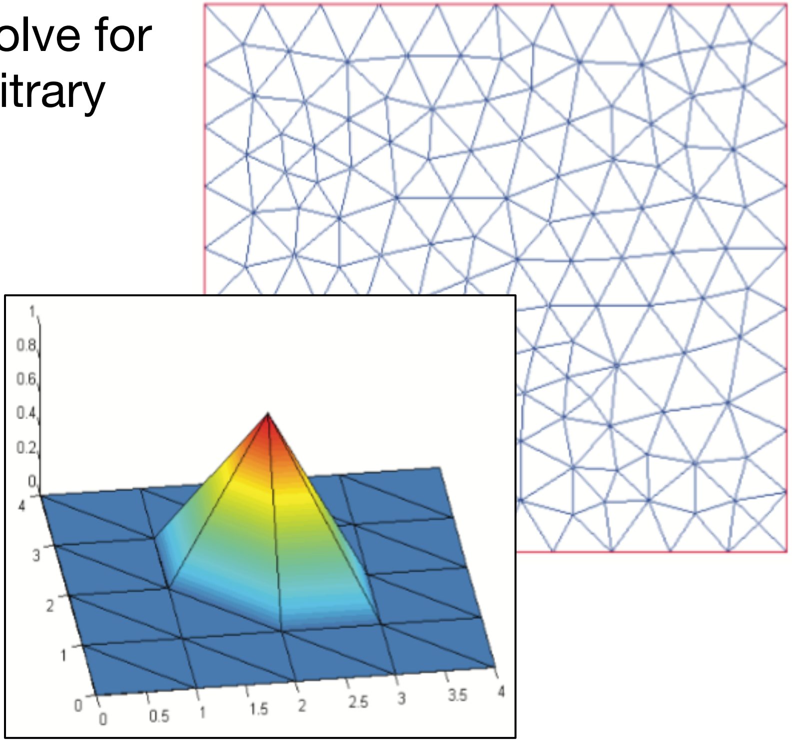
Approximation

To get an approximate solution, it makes more sense to work with the integral for $\delta\Pi$. This gives us a single number that we can use to evaluate the accuracy of our approximation:

Recall: $$\delta\Pi = \int_{B_0} \left\{ T^{1PK} : \left(\nabla_L \delta \underline{x} \right) \right\} dV_0 - \int_{B} \left\{ \frac{1}{2} \underline{D} \cdot \left(\nabla_E \delta\Phi \right) \right\} dV + \int_{\partial B} \left\{ \eta\delta\Phi - \underline{t} \cdot \delta\underline{x} \right\} dA$$

The condition $\delta\Pi = 0$ is known as the "Weak Form" of our governing equations:

(x, Φ) ~ unknown functions we need to solve for
(δx, $\delta\Phi$) ~ "weight functions" that are arbitrary



To get an approximate solution, we divide the domain into a finite # of triangular elements.

Between each node, (x, Φ, δx, $\delta\Phi$) are treated as a linear combination of prescribed basis functions ϕ_i (e.g. lines, polynomials/splines).

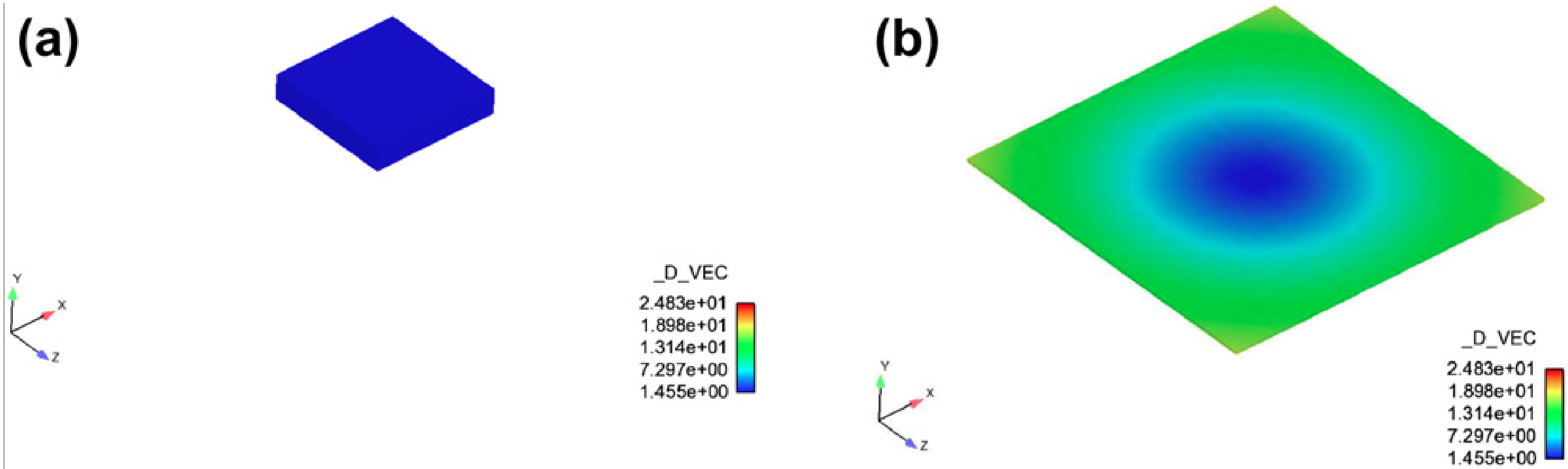
Finite Element Method

$$\underline{x} = \sum_{i=1}^{N} \underline{\alpha}_i \phi_i\left(\underline{X}\right) \quad \Phi = \sum_{i=1}^{N} \beta_i \phi_i\left(\underline{X}\right) \quad \delta\underline{x} = \sum_{i=1}^{N} \underline{\gamma}_i \phi_i\left(\underline{X}\right) \quad \delta\Phi = \sum_{i=1}^{N} \chi_i \phi_i\left(\underline{X}\right)$$

- Here, ($\underline{\alpha}_i$, β_i) are unknown and ($\underline{\gamma}_i$, ϕ_i) are arbitrary.
- When we substitute these expressions into $\delta\Pi$, the integral turns into a summation. Therefore it can be computed by performing matrix operations.
- After some matrix manipulation, the arbitrary values ($\underline{\gamma}_i$, ϕ_i) drop out. We perform a numerical root finding algorithm to solve for ($\underline{\alpha}_i$, β_i).
- For nonlinear problems, this is typically done with a gradient descent technique, e.g. Newton-Rapson, Gauss-Newton iteration:
  - Residual is linearized about a certain guess for ($\underline{\alpha}_i$, β_i)
  - Solving linearized equation leads to a new guess for ($\underline{\alpha}_i$, β_i)
  - Method will only converge to a local solution if (i) the initial guess is sufficiently close and (ii) linearized matrices are well-conditioned (i.e. non-singular).
- The accuracy of the approximation is evaluated by the convergence of the solution with increasing meshsize (N).

DEA Modeling

H. S. Park, Z. Suo, J. Zhou, P. A. Klein, "A dynamic finite element method for inhomogenous deformationand electromechanical instability of dielectric elastomer transducers," *Int. J. Solids & Struct.* **49** 2187-2194 (2012).



N_a = shape (basis) function

FEA Discretization:

$$\mathbf{x}(\mathbf{X},t) - \mathbf{X} = \sum N_a(\mathbf{X})\mathbf{u}_a(t)$$

$$\Phi(\mathbf{X},t) = \sum N_a(\mathbf{X})\Phi_a(t)$$

Weight (test) functions:

$$\xi_i(\mathbf{X}) = \sum N_a(\mathbf{X})\xi_{ia}$$

$$\eta(\mathbf{X}) = \sum N_a(\mathbf{X})\eta_a$$

Free Energy

Arruda-Boyce Model:
(Helmholtz)

$$\frac{W_{stretch}}{\mu} = \frac{1}{2}(I-3) + \frac{1}{20N}(I^2-9) + \frac{11}{1050N^2}(I^3-27) + \frac{19}{7000N^3}(I^4-81) + \frac{519}{673750N^4}(I^5-243)$$

Total Free Energy:
(Gibbs)

$$\hat{W}(\mathbf{C},\tilde{\mathbf{E}}) = W_{stretch}(I) + \overbrace{\frac{1}{2}\lambda(\log J)^2 - 2W'_{stretch}(3)\log J}^{\text{Incompressability "Penalty"}} \underbrace{- \frac{\varepsilon}{2} J C^{-1}_{IJ} \tilde{E}_I \tilde{E}_J}_{\text{Electrical Enthalpy}}$$

Lagrangian Parameter
("Hydrostatic Pressure")

1st Piola-Kirchoff Stress: $s_{iJ} = 2F_{iL}\dfrac{\partial \hat{W}(\mathbf{C},\tilde{\mathbf{E}})}{\partial C_{JL}}$

Electrical Displacement: $\tilde{D}_J = -\dfrac{\partial \hat{W}(\mathbf{C},\tilde{\mathbf{E}})}{\partial \tilde{E}_J}$

C = F^{T}F

"Right Cauchy-Green Tensor"

Governing Equation

Weak Form of PDEs:

1st Piola-Kirchoff Stress Body Force (gravity) Density Acceleration Surface traction

Stress Balance:

$$\int s_{iJ} \frac{\partial \xi_i}{\partial X_J} dV = \int \left(B_i - \rho \frac{\partial^2 x_i}{\partial t^2} \right) \xi_i dV + \int T_i \xi_i dA$$

Maxwell's Equation:

$$-\int \tilde{D}_I \frac{\partial \eta}{\partial X_I} dV = \int q\eta dV + \int \omega\eta dA$$

Electric Displacement Space Charge Surface Charge

Matrix Form:

$$\mathbf{g}(\mathbf{u}, \Phi, t) = \mathbf{Ma}$$

$$\mathbf{h}(\mathbf{u}, \Phi, t) = 0$$

Root Finding

Matrix Form:

$$\mathbf{g}(\mathbf{u}, \Phi, t) = \cancelto{\mathbf{0}\ \text{(quasistatic)}}{\mathbf{Ma}}$$

$$\mathbf{h}(\mathbf{u}, \Phi, t) = 0$$

Linearization:

$$\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{me} \\ \mathbf{K}_{me}^T & \mathbf{K}_{ee} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\Phi \end{bmatrix} = \begin{bmatrix} \mathbf{f}_m \\ \mathbf{f}_e \end{bmatrix}$$

$$\mathbf{K}_{\mathrm{mm}} = \int H_{iJkL} \frac{\partial N_a}{\partial X_J} \frac{\partial N_b}{\partial X_L} dV$$

$$\mathbf{K}_{\mathrm{me}} = -\int_V e_{kJL} \frac{\partial N_a}{\partial X_J} \frac{\partial N_b}{\partial X_L} dV$$

$$\mathbf{K}_{\mathrm{ee}} = -\int_V \varepsilon_{JL} \frac{\partial N_a}{\partial X_J} \frac{\partial N_b}{\partial X_L} dV$$

$$\mathbf{f}_{\mathrm{m}} = \int_V B_i N_a dV + \int_A T_i N_a dA - \int_V s_{iJ} \frac{\partial N_a}{\partial X_J} dV$$

$$\mathbf{f}_{\mathrm{e}} = \int_V q N_a dV + \int_A \omega N_a dA + \int_V \tilde{D}_J \frac{\partial N_a}{\partial X_J} dV$$

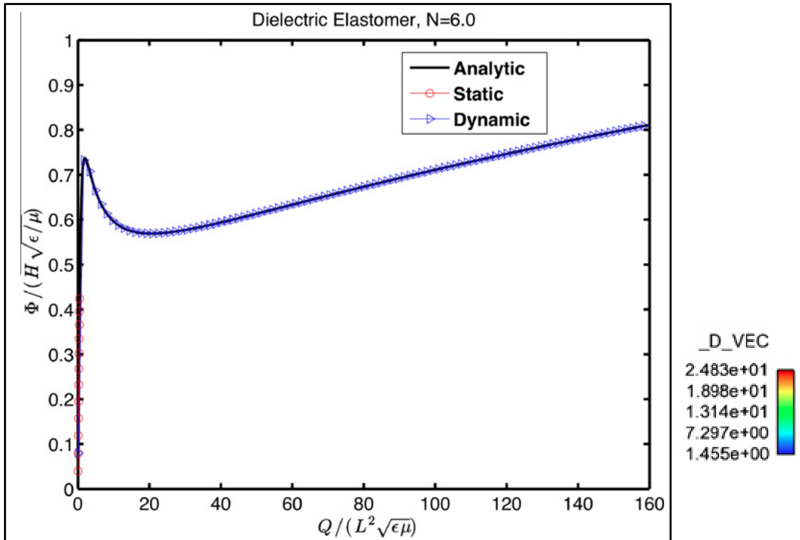
$$H_{iJkL} = 2\delta_{ik} \frac{\partial \hat{W}(\mathbf{C}, \tilde{\mathbf{E}})}{\partial C_{JL}} + 4F_{iM}F_{kN} \frac{\partial^2 \hat{W}(\mathbf{C}, \tilde{\mathbf{E}})}{\partial C_{JM} \partial C_{LN}}$$

$$e_{iJL} = -2F_{iM} \frac{\partial^2 \hat{W}(\mathbf{C}, \tilde{\mathbf{E}})}{\partial C_{JM} \partial \tilde{E}_L},$$

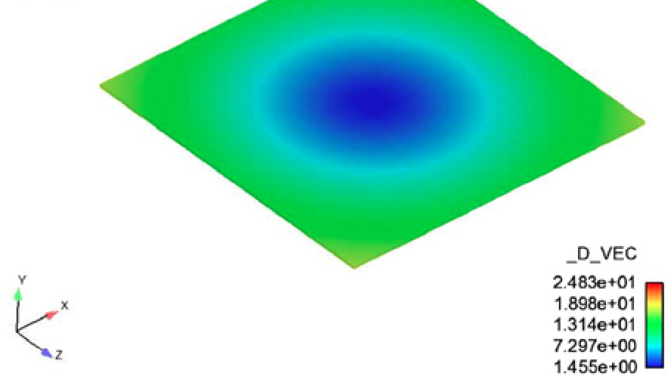
$$\varepsilon_{JL} = -\frac{\partial^2 \hat{W}(\mathbf{C}, \tilde{\mathbf{E}})}{\partial \tilde{E}_J \partial \tilde{E}_L}$$

- Find roots using Newton-Raphson method (i.e. solve linearized equation at each iteration)
- Instability when Hessian becomes singular

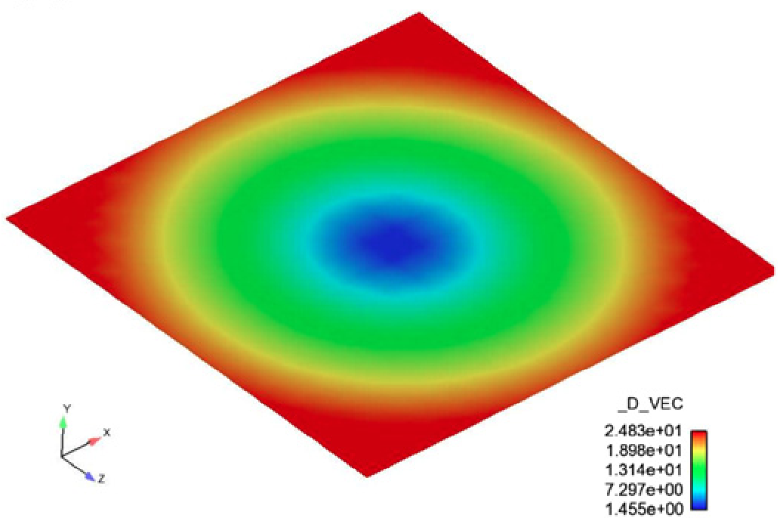
Results


Dielectric Elastomer, N=6.0
Analytic
Static
Dynamic
$\Phi/(H\sqrt{\epsilon/\mu})$
$Q/(L^2\sqrt{\epsilon\mu})$
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 20 40 60 80 100 120 140 160
_D_VEC
2.483e+01
1.898e+01
1.314e+01
7.297e+00
1.455e+00

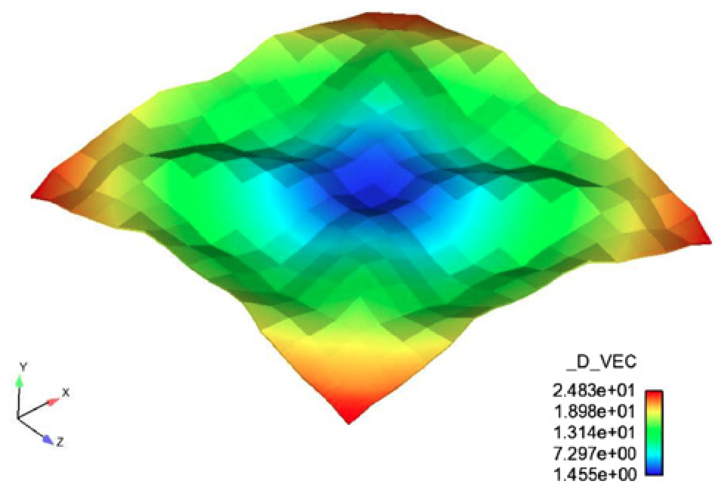
(b)


_D_VEC
2.483e+01
1.898e+01
1.314e+01
7.297e+00
1.455e+00
Y
X
Z

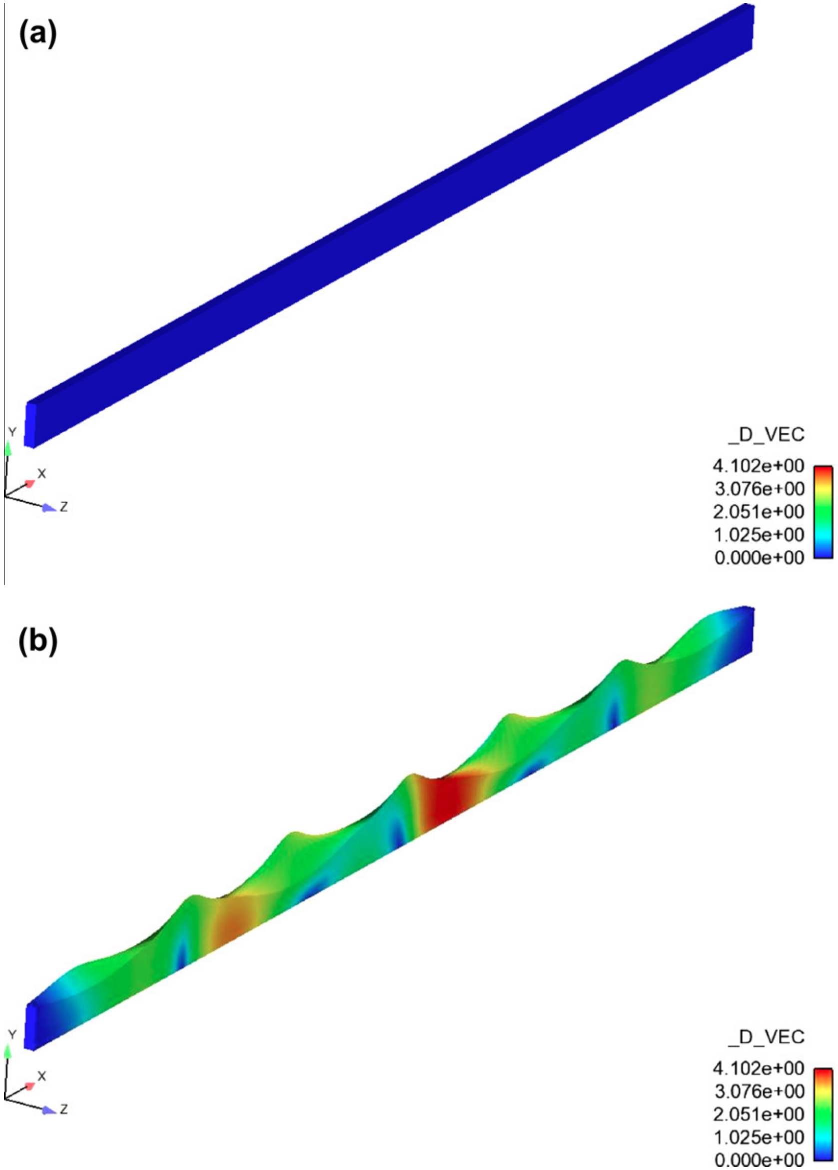
(c)


_D_VEC
2.483e+01
1.898e+01
1.314e+01
7.297e+00
1.455e+00
Y
X
Z

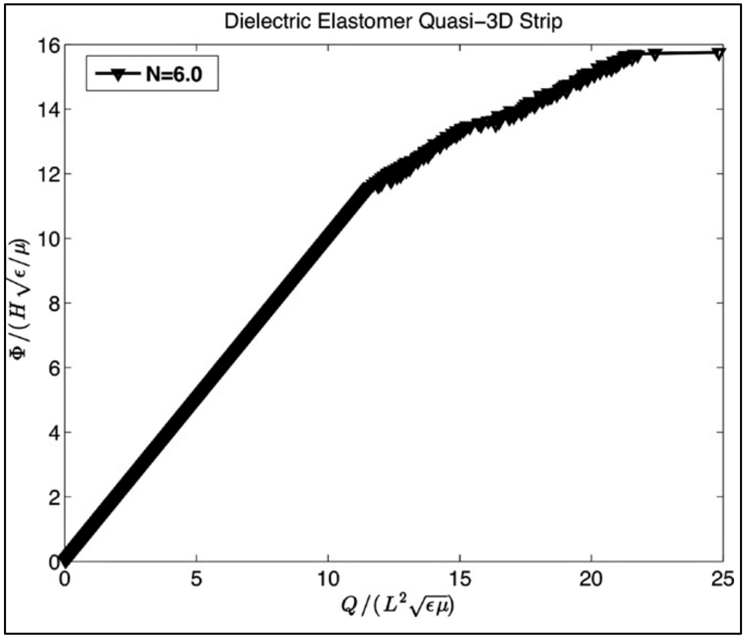
(d)


_D_VEC
2.483e+01
1.898e+01
1.314e+01
7.297e+00
1.455e+00
Y
X
Z

Results



(a)
Y
X
Z
_D_VEC
4.102e+00
3.076e+00
2.051e+00
1.025e+00
0.000e+00
(b)
Y
X
Z
_D_VEC
4.102e+00
3.076e+00
2.051e+00
1.025e+00
0.000e+00



Dielectric Elastomer Quasi-3D Strip
N=6.0
$\Phi/(H\sqrt{\epsilon/\mu})$
$Q/(L^2\sqrt{\epsilon\mu})$
0
2
4
6
8
10
12
14
16
0
5
10
15
20
25

CAD MODELING – ISSUES

No algebraic solution

- Scaling laws and design rules?
- How will changing a material property or geometric dimension alter performance?

Takes too long

- Computationally intensive
- Requires hours to run a simulation
- Simulation must be repeated for even minor design changes
- Solution doesn’t always converge

Difficult to validate

- No method to independenty validate solution
- Dependent on accuracy of assumptions/input
- “Garbage in, garbage out”

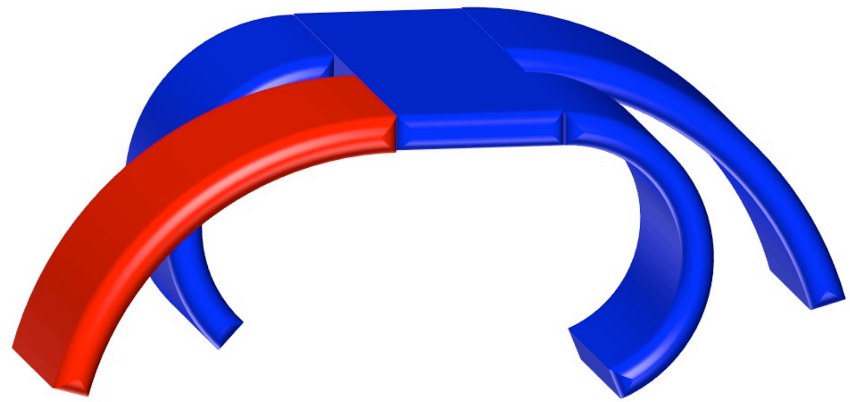
PROPOSED ALTERNATIVE

Discretize!

- Treat each limb as an individual element

Use “Reduce Dimensional” Models

- Euler-Bernoulli Beam Theory
- Kirchoff Plate Theory
- Coulomb’s Friction Law
- Hertzian Contact Theory



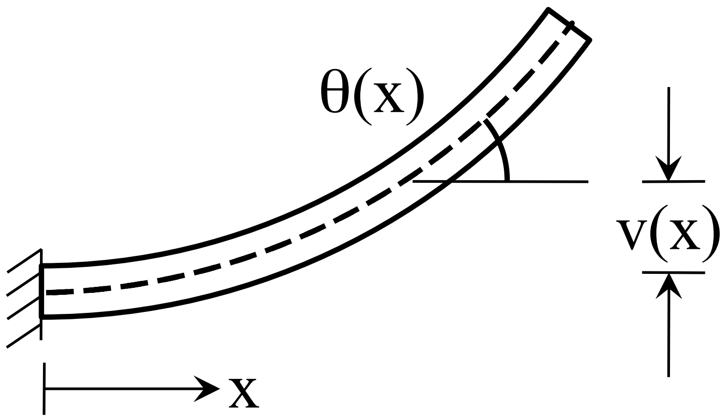
Ensure Compatibility

- Attached elements must be *kinematically compatible*
- Transfer of equal-and-opposite loads

Use ODE solvers in MATLAB

- Runga-Kutta (ode45)
- Finite difference (bvp4c)
- Avoid PDEs whenever possible!

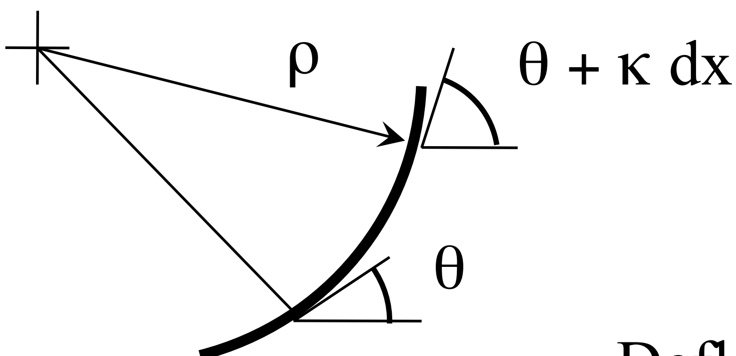
EULER-BERNOULLI BEAM THEORY



v = deflection

$$\theta = \frac{dv}{dx} = \text{slope}$$

$$\kappa = \frac{d\theta}{dx} = \frac{1}{\rho} = \text{curvature}$$

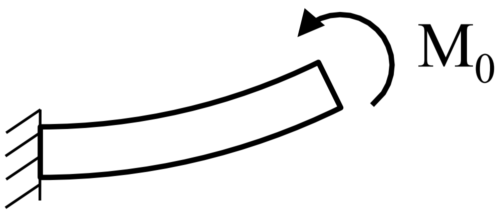


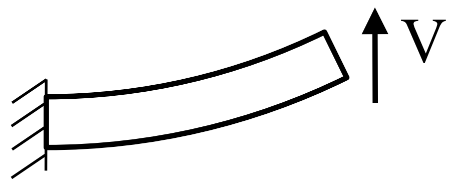
ρ = radius of curvature

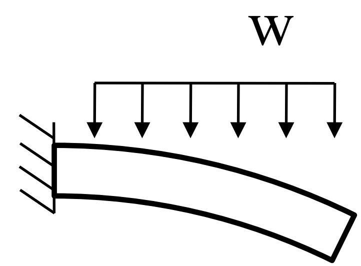
$$\boxed{\kappa = \frac{d\theta}{dx^2} = \frac{d^2v}{dx^2} \equiv \frac{M}{EI}}$$

Deflection are determined by calculating the internal bending moment $M = M(x)$.

Examples







$$M = M_0 \qquad M = V(L - x) \qquad M = -w(L - x)^2/2$$

$$\frac{d^2v}{dx^2} = \frac{M_0}{EI} \qquad \frac{d^2v}{dx^2} = \frac{V}{EI}\left(L - x\right) \qquad \frac{d^2v}{dx^2} = -\frac{w}{2EI}\left(L - x\right)^2$$

$$v(0) = \left.\frac{dv}{dx}\right|_{x=0} = 0 \qquad v(0) = \left.\frac{dv}{dx}\right|_{x=0} = 0 \qquad v(0) = \left.\frac{dv}{dx}\right|_{x=0} = 0$$

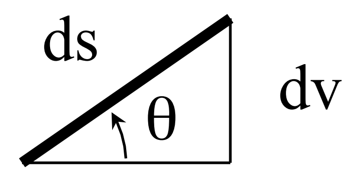
ELASTICA

For large deflections, use Elastica theory

In both Elastica and Linear Beam Theory, $M = D\kappa$, where

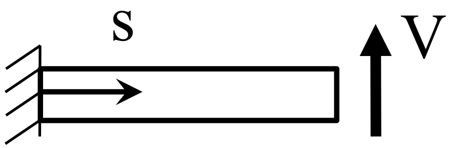
$$M = M(\xi) \qquad \text{and} \qquad \kappa = d\theta/ds$$

The difference between the two theories is how we calculate M and θ.



Linear Beam Theory	Elastica
$dv/ds = \theta$	$dv/ds = \sin(\theta)$
M calculated in Lagrangian Description (Ref. Placement)	M calculated in Eulerian Description (Current)

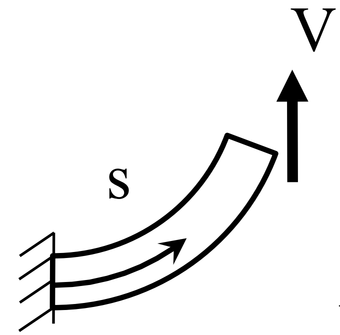
<u>Linear Beam Theory</u>

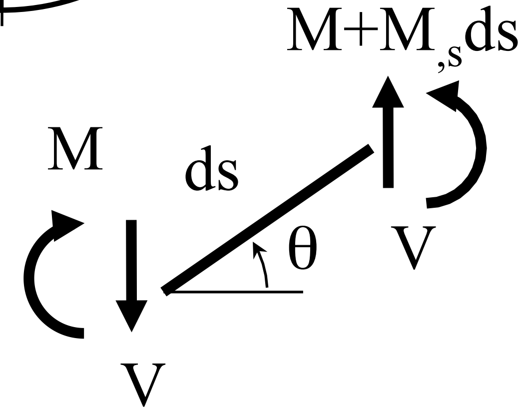


$$M = V\left(L - s\right)$$

$$\frac{d\theta}{ds} = \frac{V}{D}\left(L - s\right)$$

<u>Elastica</u>





$$\frac{dM}{ds} = -V\cos\left(\theta\right)$$

$$\frac{d^2\theta}{ds^2} = -\frac{V}{D}\cos\left(\theta\right)$$

Converge for small deflection: $\cos(\theta) \approx 1$

$dM/ds \approx -V \Rightarrow M = C_1 - Vs$

$M(L) = 0 \;\Rightarrow C_1 = Ls$

$\Rightarrow d\theta/ds = -(V/D)(L - s)$

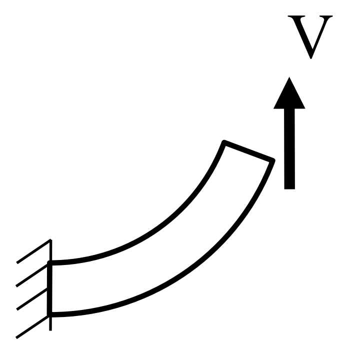
In general, $\dfrac{d^2\theta}{ds^2} = k_1 \sin(\theta) + k_2 \cos(\theta)$

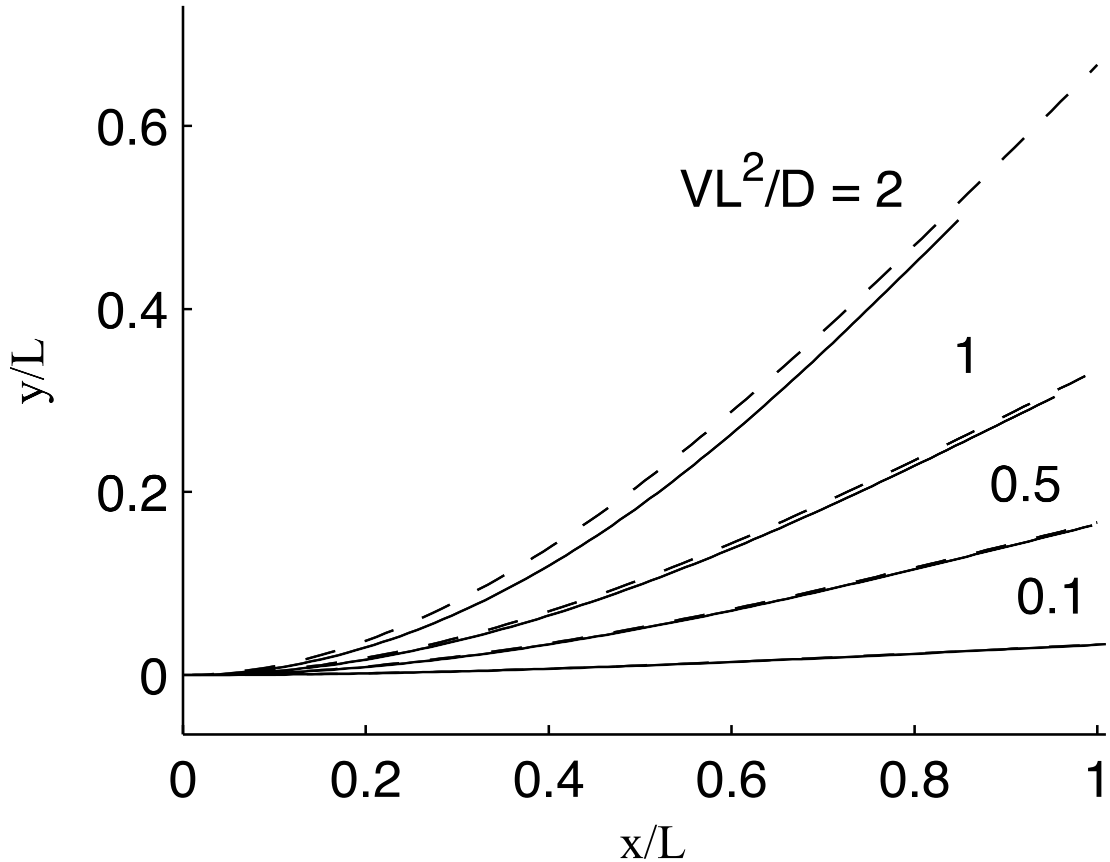
(non-linear 2^{nd}-order ODE)

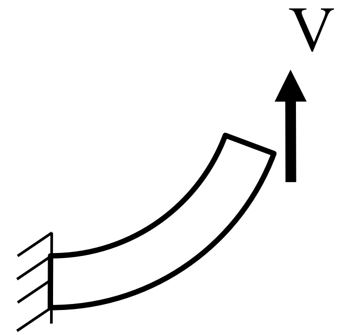
Must solve for $\theta(0) = 0$ and $\theta'(L) = M_L$

For pure shear, is determined by solving the following boundary-value problem (BVP):

$$\frac{d^2\theta}{ds^2} = -\frac{V}{D}\cos(\theta) \quad \theta(0) = 0 \quad \theta'(L) = 0$$







solid = Elastica

dashed = Linear Beam Theory →

$$\begin{cases} x = s \\ y = \dfrac{Vs^2}{6D}\left(3L - s\right) \end{cases}$$

We typically solve nonlinear BVPs in Matlab using bvp4c:

```
function elastica

global D V R0

n = 100;
L = 1;
D = 1;
V = 2;

s = linspace(0,L,n);
ds = L/(n-1);

solinit = bvpinit(s,@axial_init);
sol = bvp4c(@axial_ode,@axial_bc,solinit);

S = deval(sol,s);
theta = S(1,:);
kappa = S(2,:);
```

$$\frac{d^2\theta}{ds^2} = -\frac{V}{D}\cos(\theta) \quad \theta(0) = 0 \quad \theta'(L) = 0$$

Matlab solves 1st order ODEs (scalars or vectors). Convert:

$$z = \begin{pmatrix} \theta \\ \theta' \end{pmatrix} \Rightarrow z' = \begin{pmatrix} z_2 \\ -\frac{V}{D}\cos(z_1) \end{pmatrix}$$

$$z_1(0) = 0 \quad z_2(L) = 0$$

```
solinit = bvpinit(s,@axial_init);
sol = bvp4c(@axial_ode,@axial_bc,solinit);
...
% ------------------------------------
function dzdr = axial_ode(s,z)

global D V
dzdr = [  z(2); -(V/D)*cos(z(1))];

% ------------------------------------
function res = axial_bc(z0,zL)

res = [ z0(1);  zL(2)];

% ------------------------------------
function yinit = axial_init(s)

yinit = [  0;     0 ];
```

$$\frac{d^2\theta}{ds^2} = -\frac{V}{D}\cos(\theta)$$

$$\theta(0) = 0 \quad \theta'(L) = 0$$

```
S = deval(sol,s);
theta = S(1,:);
kappa = S(2,:);

x = tril(ones(n,n))*(cos(theta))'*ds;
y = tril(ones(n,n))*(sin(theta))'*ds;

figure(1)
hold on
plot(x,y,'k-')
```

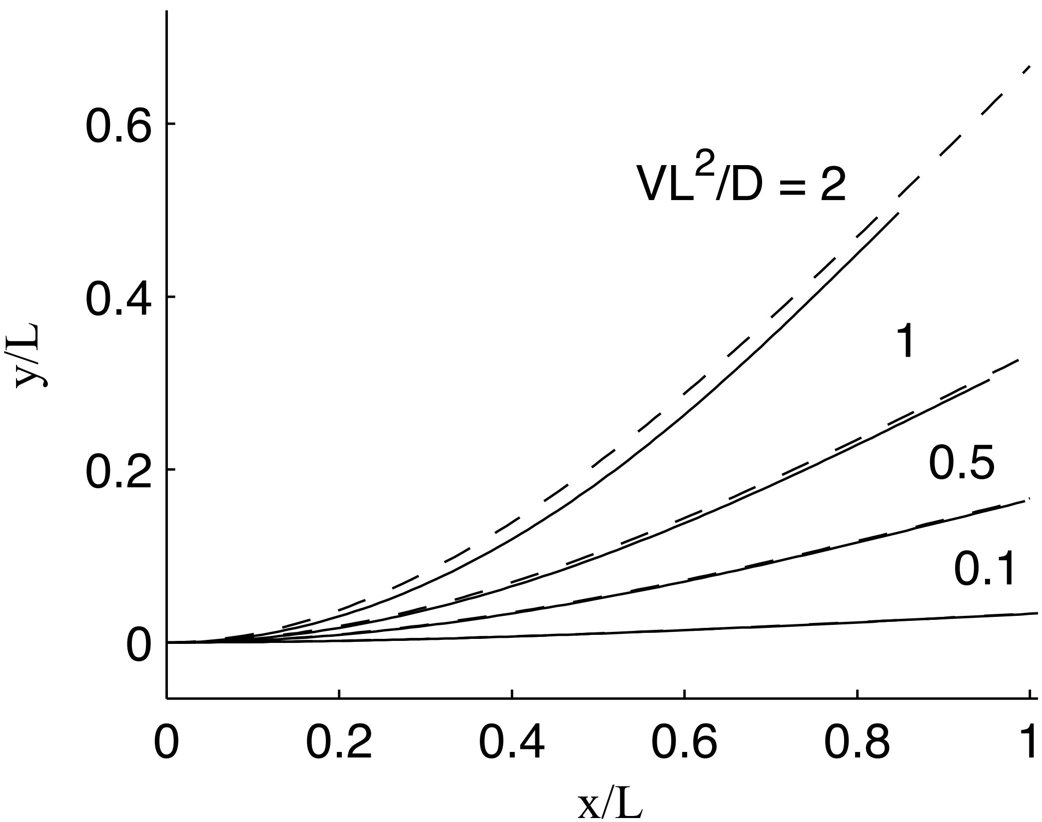
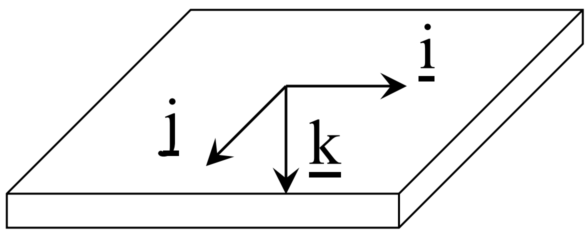
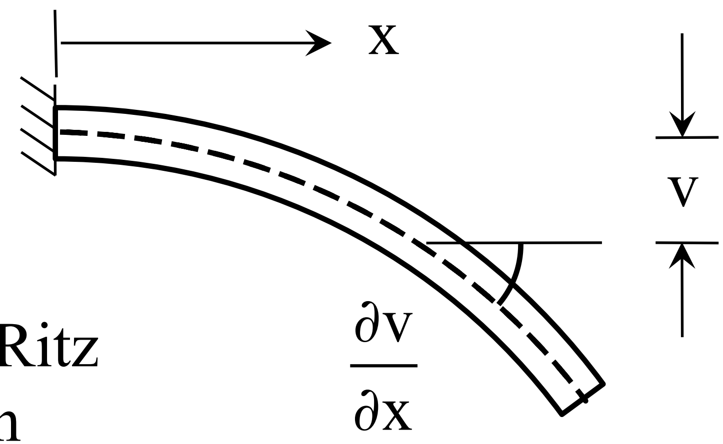


PLATE THEORY

Deflection: $v = v(x, y)$

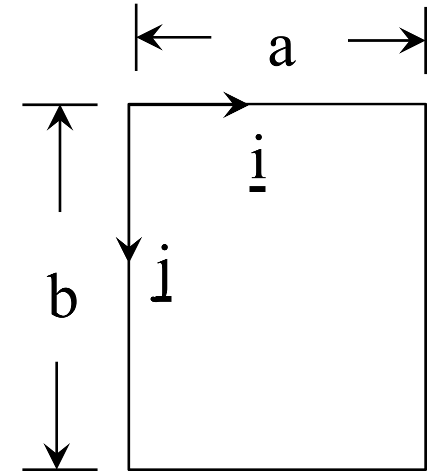




Estimate deflection using Rayleigh-Ritz method. For a rectangular plate with weight per unit area p:

$$\Pi = \iint \left\{ \frac{D}{2} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)^2 - vp \right\} dxdy$$

Suppose that a rectangular plate with simple supports and edges of length a and b is uniformly loaded. To obtain an estimate for v = v(x,y) , we will assume that it has the form



$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

This implies

$$\Pi = \int_0^a\int_0^b \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{D}{2}\left[a_{mn}\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\right]^2 - p_0\, a_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\right\}dxdy$$

Integrating, it follows that

$$\Pi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\pi^4 abD}{8}a_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{4p_0 ab}{\pi^2 mn}a_{mn}\right\}$$

for m,n = 1, 3, …

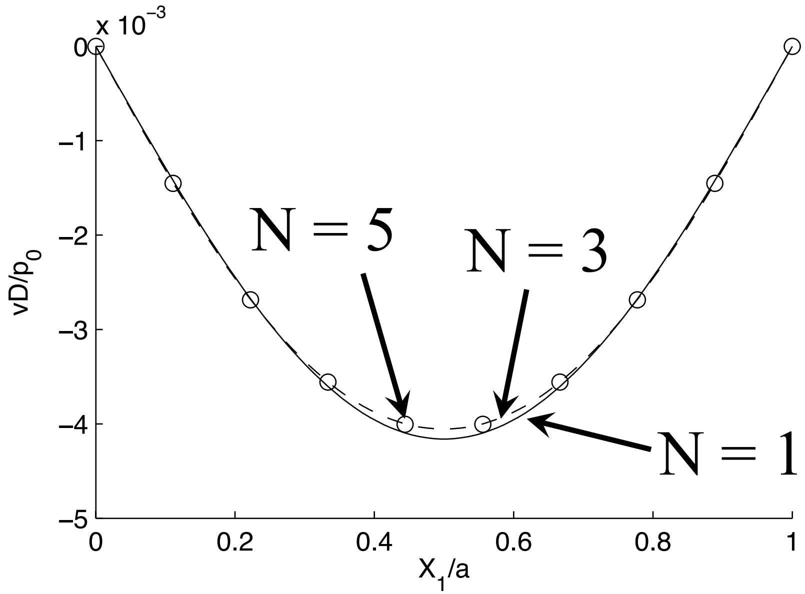
At equilibrium, $\dfrac{d\Pi}{da_{mn}} = 0$

Solving for a_{mn} yields

$$a_{mn} = \frac{16p_0}{\pi^6 mnD\left[(m/a)^2+(n/b)^2\right]^2}$$

$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16p_0}{\pi^6 mnD\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]^2} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

for m,n = 1, 3, …

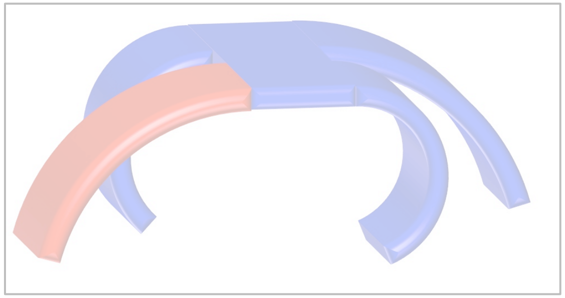


Review of actuator technologies

Pneumatic • DEA • SMA • IPMC • Bio-hybrid

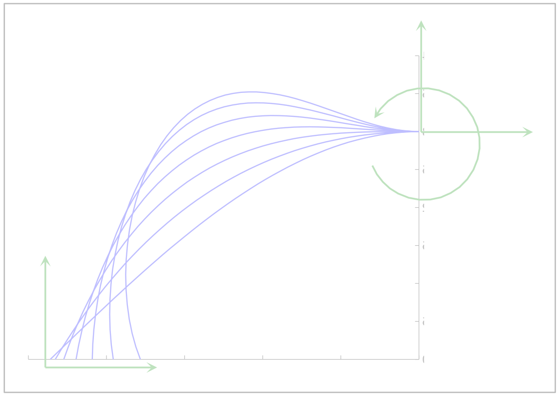
Representations of Soft Robot Limbs

Euler-Bernouli Beam • Elastica • Plate/Shell

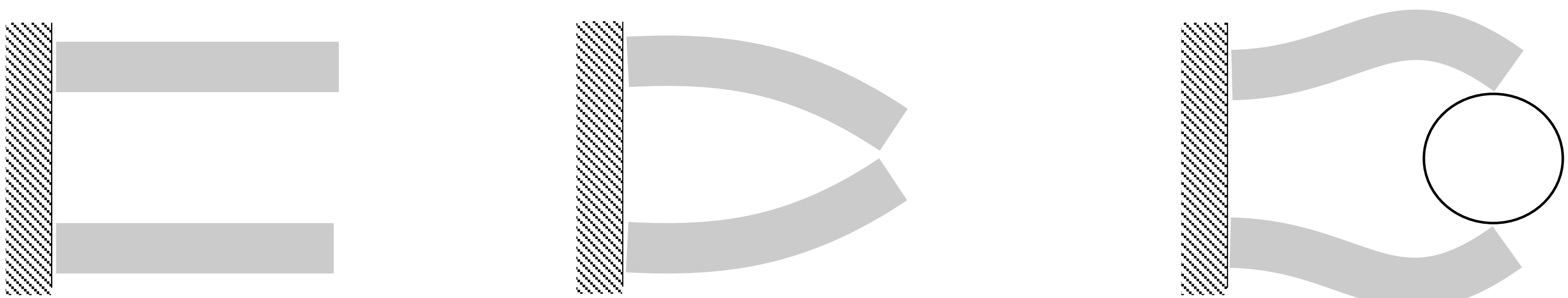


Simulation of Soft Robot Limbs

Kinematics • Tribology • Analysis (ODE/PDE)



SOFT ROBOT GRIPPER



open: $(\kappa_0)_{open} = 0$ closed: $(\kappa_0)_{closed} = \kappa_0 > 0$

Deflection: $v = v(x)$

- What is $v(x)$?
- What is the contact force F?
- What is the gripping strength?

SOFT ROBOT GRIPPER

$$m = D\left(v'' - \kappa_0\right)$$

$$m' = F \Rightarrow v''' = \frac{F}{D}$$

BCs

$$v(0) = v(L) = v'(0) = 0$$

$$m(L) = 0 \Rightarrow v''(L) = \kappa_0$$

LINEAR BEAM THEORY

$$v''' = \frac{F}{D} \Rightarrow v = c_0 + c_1 x + c_2 x^2 + \frac{F}{6D} x^3$$

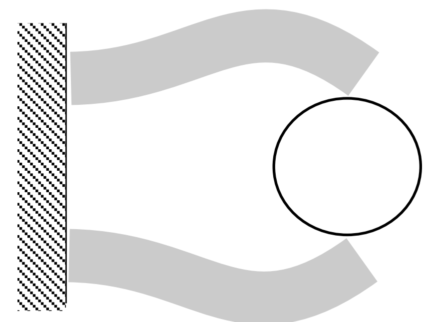
$$v(0) = v'(0) = 0 \Rightarrow v = c_2 x^2 + \frac{F}{6D} x^3$$

$$v(L) = 0 \Rightarrow c_2 = -\frac{FL}{6D}$$

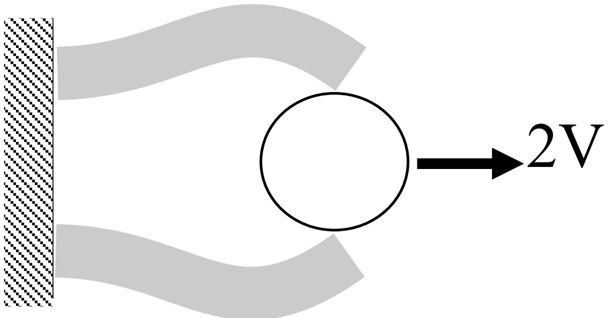
$$v = \frac{F}{6D}\left(x^3 - x^2 L\right)$$

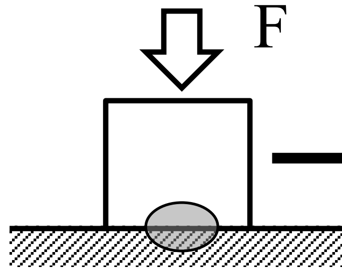
$$v''(L) = \kappa_0 \Rightarrow \boxed{F = \frac{3D\kappa_0}{2L}}$$

$$\therefore \boxed{v = \frac{\kappa_0 x^2}{4}\left(\frac{x}{L} - 1\right)}$$



COULOMB'S LAW





$$V = \underbrace{\mu_0 F}_{\text{Amonton's Law}} + \underbrace{\tau A_t}_{\text{Adhesion-Controlled Friction}}$$

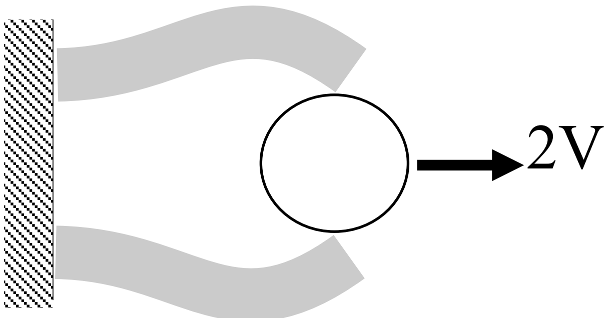
$\mu_0 F$ ~ Mechanical sliding resistance of interlocking asperities

τA_t ~ interfacial shear strength

According to Contact Mechanics $A_t \approx A_0 + \alpha F$

e.g. Greenwood-Williamson & Johnson-Kendall-Roberts Theories

GRIPPING STRENGTH



$$V = \mu_0 F + \tau(\alpha F + A_0)$$

$$= \mu F + V_0$$

where $\mu = \mu_0 + \tau\alpha$

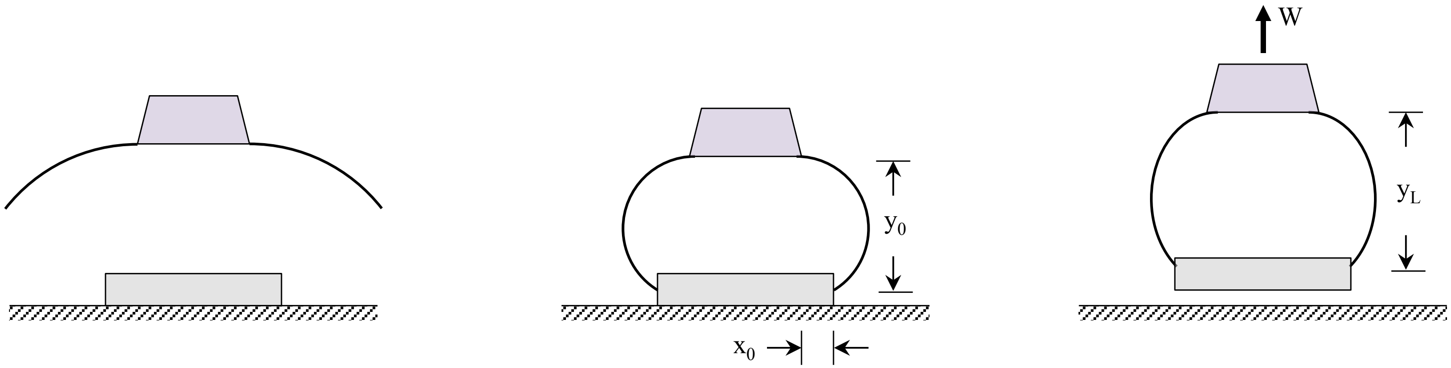
and $V_0 = \tau A_0$

Ignoring the initial adhesion (i.e. $V_0 \approx 0$) slip occurs when $\boxed{V > \mu F = \dfrac{3\mu D\kappa_0}{2L}}$

Limitations of Linearized theory:

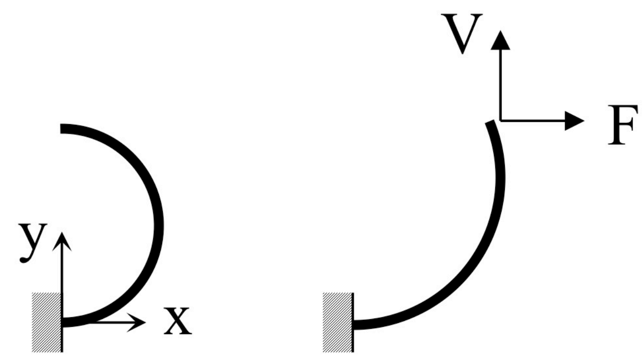
- Kinematics not accurate for large deflection
- Does not account for influence of axial load during pick & place operations

Two Finger Gripper



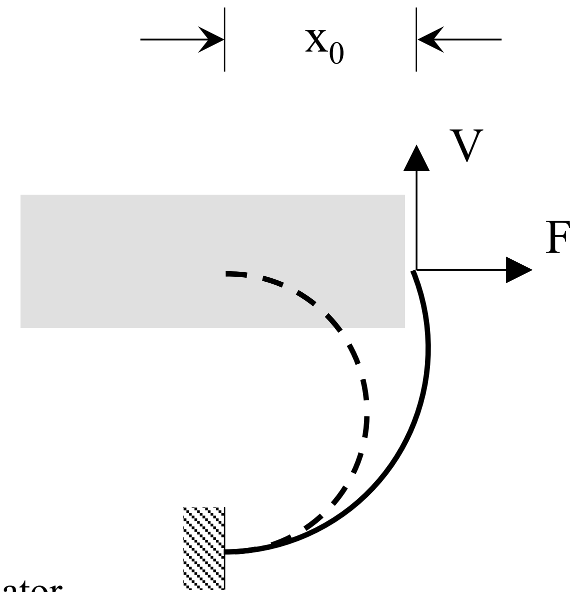
Finger ~ Naturally Curved Elastic rod

- Length L, curvature κ_0
- Natural curvature controlled by actuator
  - open: $(\kappa_0)_{open}$
  - closed: $(\kappa_0)_{closed}$
- Flexural rigidity D = EI
  - E = Young's Modulus
  - I = $wh^3/12$ = Area Moment of Inertia
- Fixed slope at base
- Contact loads F and V at the tip
  - F = normal reaction force to prevent interpenetration
  - V = tangential frictional resistance to sliding
- Large deflection bending
  - Small angle approximation and Euler-Bernoulli beam theory are *not valid*
  - Use Elastica theory – planar bending; large angle deflection; small bending strains



HW 5

1 Flexible Bending Actuator. A flexible bending actuator can be treated as an inextensible rod with a natural bending curvature κ_0 and flexural rigidity D. Let L = 5 cm, $D = 4\times10^{-6}$ Nm^2, $\kappa_0 = 60$ m^{-1}.

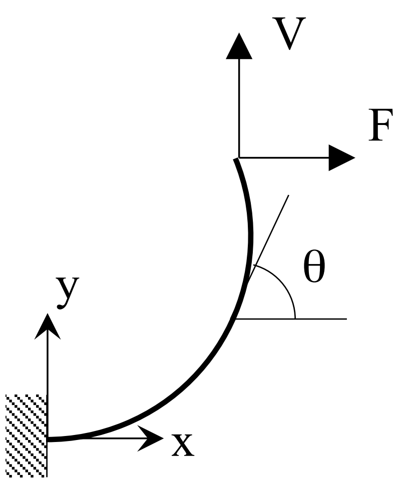


Suppose that point loads F and V are applied to the free end, as shown. The slope $\theta = \theta(s)$ is determined by minimizing the functional

$$\Pi = \Pi(\kappa) = \int_0^L \frac{1}{2} D_{eq} (\kappa - \kappa_0)^2 ds - Fx_L - Vy_L,$$

$$\kappa = d\theta/ds = \theta'$$

(x_L, y_L) are the coordinates of the end of the actuator



Step 1: Find $\theta = \theta(s; F, V)$

Step 2: Find $x = x(s)$ and $y = y(s)$

Step 3: Find F such that $x_L := x(L)$ is equal to x_0

Step 4: Calculate maximum frictional resistance μF. If $\mu F > V$ then contact will slip

Step 1: Find θ = θ(s; F, V)

$$\Pi = \Pi(\kappa) = \int_0^L \frac{1}{2} D_{eq} (\kappa - \kappa_0)^2 ds - Fx_L - Vy_L$$

$$x_L = \int_0^L \cos\theta\, ds \quad \text{and} \quad y_L = \int_0^L \sin\theta\, ds$$

Determine θ that minimizes Π.

$$\Pi = \int_0^L \left\{ \frac{1}{2} D(\theta' - \kappa_0)^2 - F\cos\theta - V\sin\theta \right\} ds = \int_0^L \Gamma(\theta, \theta')\, ds$$

Calculus of Variations:

$$\delta\Pi = \int_0^L \left\{ \frac{\partial\Gamma}{\partial\theta}\delta\theta + \frac{\partial\Gamma}{\partial\theta'}\delta\theta' \right\} ds \equiv 0$$

$$= \int_0^L \left\{ \frac{\partial\Gamma}{\partial\theta}\delta\theta + \frac{d}{ds}\left(\frac{\partial\Gamma}{\partial\theta'}\delta\theta \right) - \left[\frac{d}{ds}\left(\frac{\partial\Gamma}{\partial\theta'} \right) \right] \delta\theta \right\} ds \equiv 0$$

$$= \left(\frac{\partial\Gamma}{\partial\theta'} \right)_{s=L} \delta\theta(L) - \left(\frac{\partial\Gamma}{\partial\theta'} \right)_{s=0} \delta\theta(0) + \int_0^L \left\{ \frac{\partial\Gamma}{\partial\theta} - \frac{d}{ds}\left(\frac{\partial\Gamma}{\partial\theta'} \right) \right\} \delta\theta\, ds \equiv 0$$

Step 1: Find θ = θ(s; F, V)

$$\boxed{\theta(0)=0} \Rightarrow \delta\theta(0)=0$$

$$\Rightarrow \delta\Pi = \left(\frac{\partial\Gamma}{\partial\theta'}\right)_{s=L}\delta\theta(L) + \int_0^L \left\{\frac{\partial\Gamma}{\partial\theta} - \frac{d}{ds}\left(\frac{\partial\Gamma}{\partial\theta'}\right)\right\}\delta\theta\, ds$$

$$\delta\Pi = 0\,\forall\delta\theta \Rightarrow \begin{cases} \left(\dfrac{\partial\Gamma}{\partial\theta'}\right)_{s=L} = 0 \\ \dfrac{\partial\Gamma}{\partial\theta} - \dfrac{d}{ds}\left(\dfrac{\partial\Gamma}{\partial\theta'}\right) = 0 \end{cases}$$

$$\frac{\partial\Gamma}{\partial\theta} = F\sin\theta - V\cos\theta$$

$$\frac{\partial\Gamma}{\partial\theta'} = D(\theta' - \kappa_0)$$

$$\left(\frac{\partial\Gamma}{\partial\theta'}\right)_{s=L} = 0 \Rightarrow D\{\theta'(L) - \kappa_0\} = 0 \qquad \boxed{\therefore \theta'(L) = \kappa_0}$$

$$\frac{\partial\Gamma}{\partial\theta} - \frac{d}{ds}\left(\frac{\partial\Gamma}{\partial\theta'}\right) = 0 \Rightarrow F\sin\theta - V\cos\theta - D\theta'' = 0$$

$$\boxed{\therefore \theta'' - \frac{F}{D}\sin\theta + \frac{V}{D}\cos\theta = 0}$$

Boundary Value Problem

$$\theta'' - \frac{F}{D}\sin\theta + \frac{V}{D}\cos\theta = 0 \quad \theta(0) = 0 \quad \theta'(L) = \kappa_0$$

In MATLAB, use bvp4c to solve θ for $0 \le s \le L$:

Step 1a: Define system parameters & variables – F, V, D, κ_0, L, s

Step 1b: Guess Solution

```
solinit = bvpinit(s,@mat4init);
%-------------------------------
function yinit = mat4init(s)
yinit = [ kappa0*s; kappa0];
```

Step 1c: Define ODE & BCs

let $z = (\theta\ \theta')$ s.t. $z_1' = z_2$ and $z_2' = (F/D)\sin(z_1) - (V/D)\sin(z_1)$

```
%-------------------------------
function dzds = mat4ode(s,z)
dzds = [  z(2); (F/D)*sin(z(1)) - (V/D)*cos(z(1)) ];
%-------------------------------
function res = mat4bc(za,zb)
res = [  za(1); zb(2)-kappa0 ];
```

Boundary Value Problem

Step 1d: Solve for z

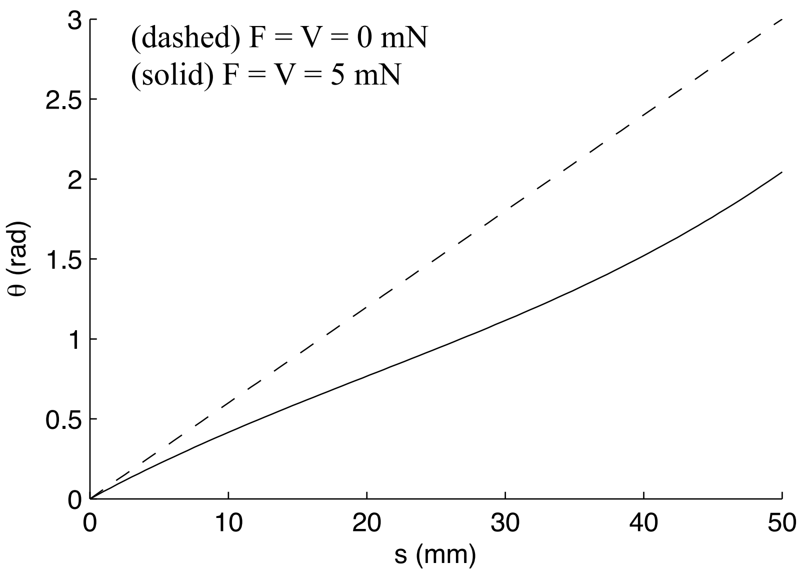
```
sol = bvp4c(@mat4ode,@mat4bc,solinit);
```

Step 1e: Obtain θ

```
z = deval(sol,s);
theta = z(1,:);
```

Step 1f: Plot θ vs. s

```
figure(1); hold on
plot(s*1e3,theta,'k-')
xlabel('s (mm)')
ylabel('\theta (rad)')
```



*Use `global` to pass system parameters (i.e. F, V, L, …) between functions

Step 2: Find x = x(s) and y = y(s)

Given θ = θ (s), we can find x and y by integration:

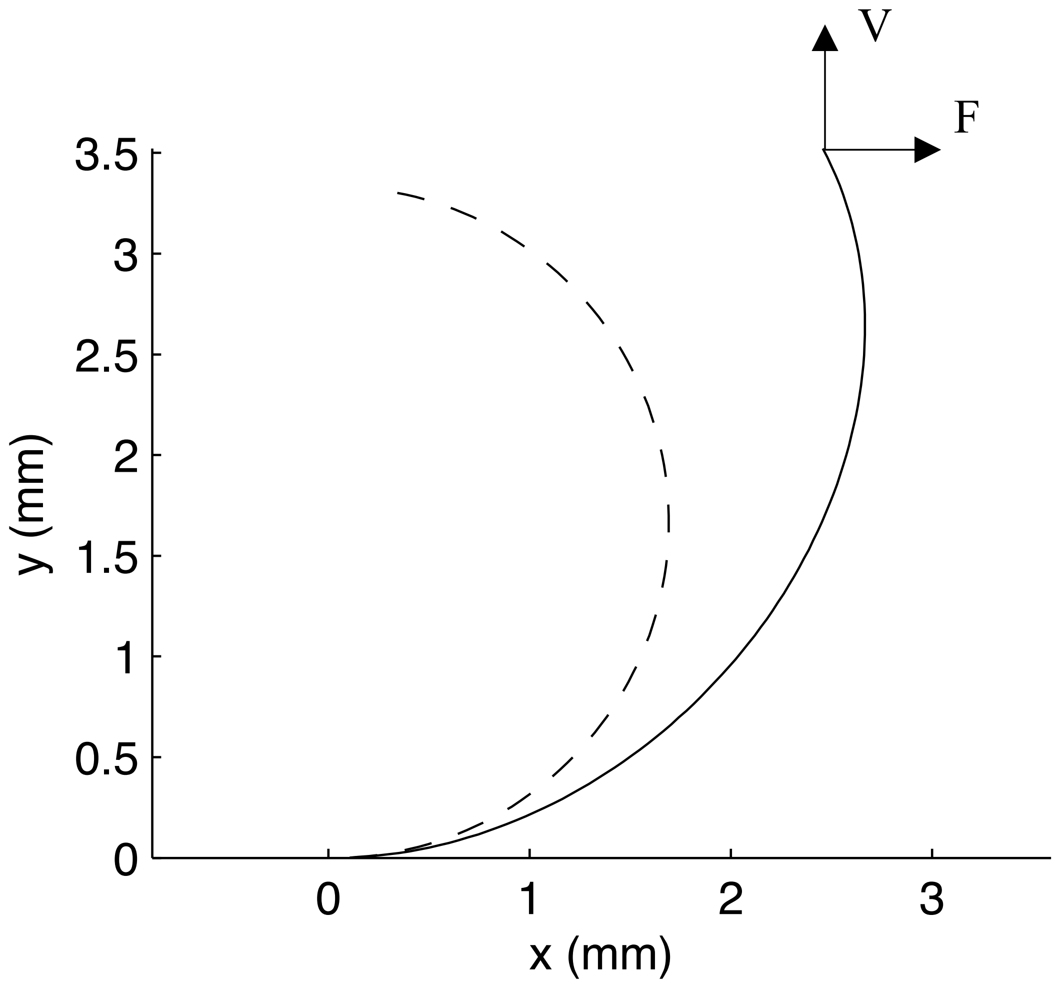
$$x(s) = \int_0^s \cos\theta(\hat{s})\,d\hat{s} \quad \text{and} \quad y(s) = \int_0^s \sin\theta(\hat{s})\,d\hat{s}$$

Shortcut: Let `s = linspace(0,L,n)` and `ds = L/(n - 1)`

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ ds \\ 2ds \\ \vdots \\ (n-1)ds = L \end{Bmatrix} \qquad \begin{matrix} x(s_1) = 0 \\ \begin{Bmatrix} x(s_2) \\ x(s_2) \\ \vdots \\ x(s_n) \end{Bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{Bmatrix} \cos\theta(s_1) \\ \cos\theta(s_2) \\ \vdots \\ \cos\theta(s_{n-1}) \end{Bmatrix} ds$$

```
x = [0; tril(ones(n-1,n-1))*cos(theta(1:n-1))'*ds];
y = [0; tril(ones(n-1,n-1))*sin(theta(1:n-1))'*ds];
```

Step 2: Find x = x(s) and y = y(s)



Step 3: Find F such that $x_L := x(L)$ is equal to x_0

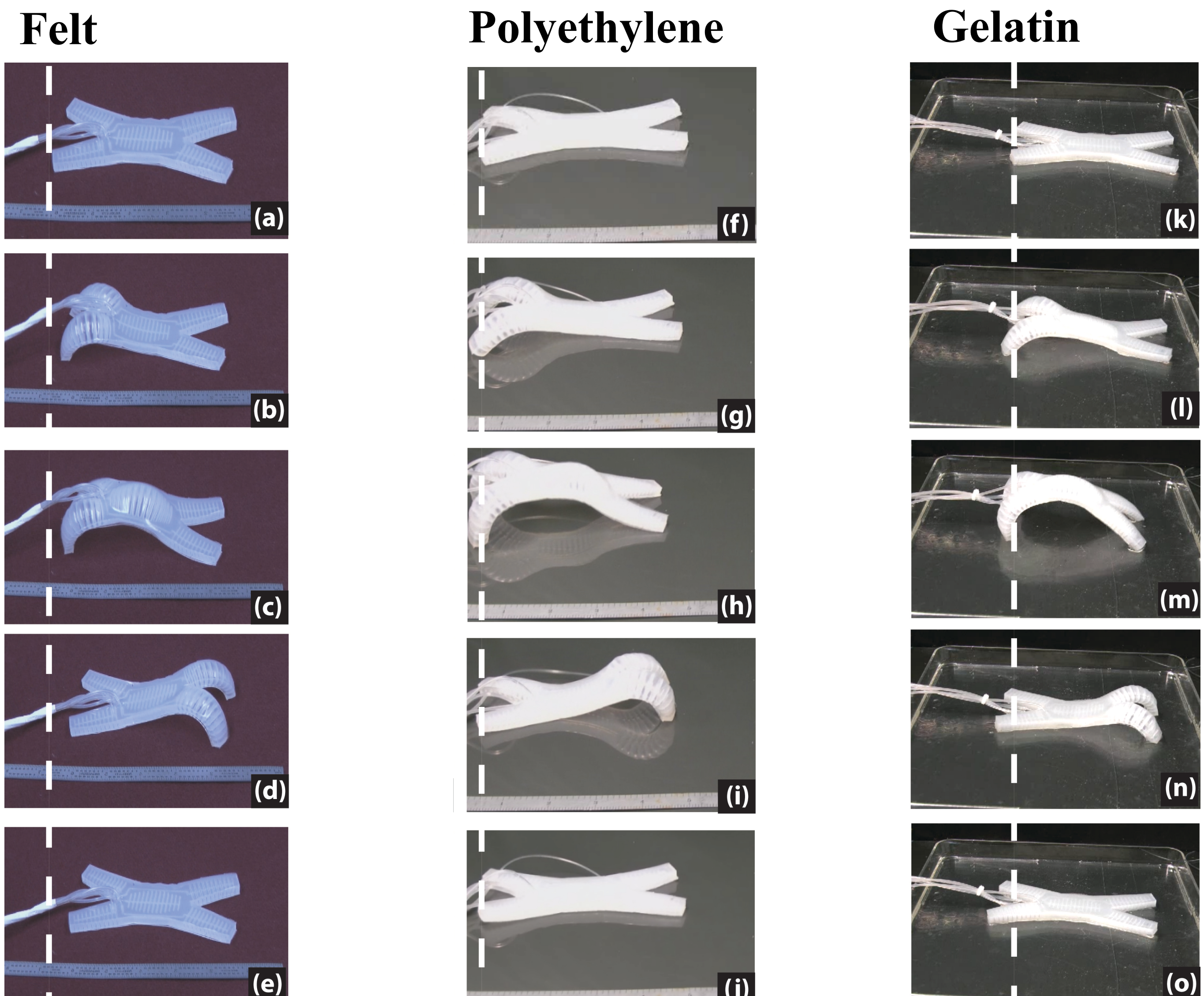
$$x_L = \int_0^L \cos\theta \, ds \equiv x_0$$

In MATLAB, use fzero:

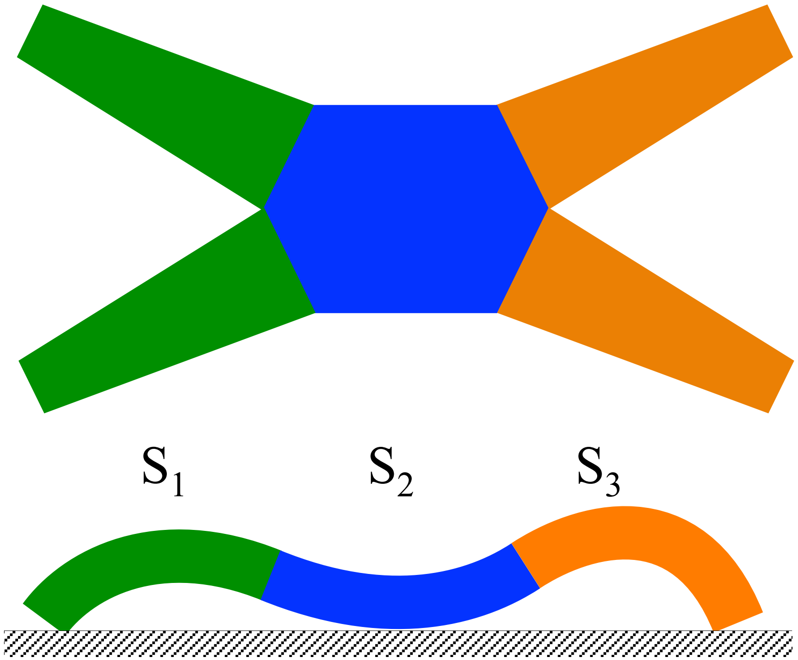
```
…
F_guess = 1e-3;
F = fzero(@get_x0,F_guess);
% -------------------------
function res = get_x0(f)
…
xL = sum(cos(theta))*ds;
res = xL - x0;
```

Step 4: Calculate maximum frictional resistance μF.
If $\mu F > V$ then contact will slip

3-SEGMENT UNDULATING ROBOT



3-Segment Undulating Robot (2D Model)

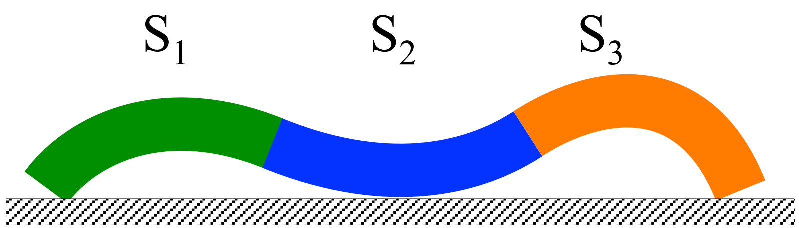


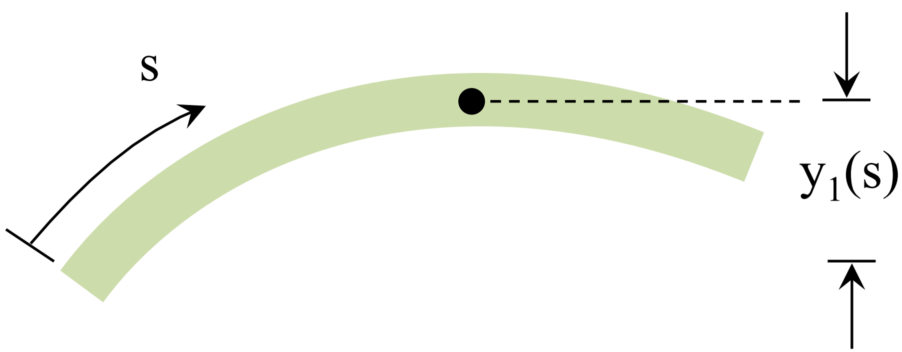
2 Limb Pairs & 1 Torso
→ 3 Segments

Segments: S_1, S_2, S_3

- Lengths L_i
- s = arclength (left to right)
- $L = L_1 + L_2 + L_3$
- $S_1 = [0, L_1)$
- $S_2 = [L_1, L_1 + L_2)$
- $S_3 = [L_1 + L_2, L]$

3-Segment Undulating Robot (2D Model)

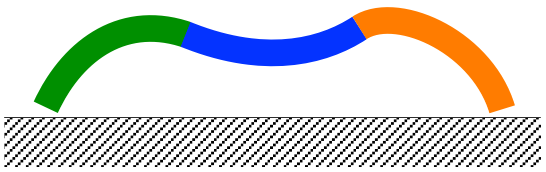




$$y_i'''' = -\frac{w}{D_i} \Rightarrow y_i = -\frac{ws^4}{24D_i} + \frac{a_i s^3}{6} + \frac{b_i s^2}{2} + c_i s + d_i$$

- Elastic with tunable flexural rigidity $D_i = D(p_i)$ and natural curvature $\kappa_i = \kappa(p_i)$
  - p_i = signal input (i.e. pressure, voltage, current, …)
  - $y_i = y_i(s)$ vertical deflection
  - $m_i = D(y_i'' - \kappa_i)$ = internal bending moment of i^{th} segment
- Gravitational loading per unit length: $w = (\rho wt)g$
  - ρ = mass density
  - w = limb width
  - t = limb thickness
  - g = gravity

Step 1: At the start of each time step, assume point contact at the ends and calculate the solution $\{a_i, b_i, c_i, d_i\}$ for the following BCs



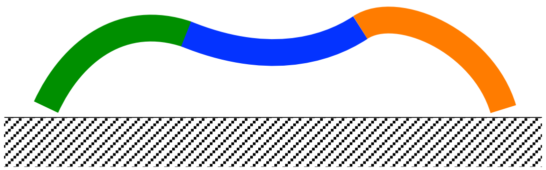
Kinematic

$$y_1(0) = y_3(L) = 0$$
$$y_1(L_1) = y_2(L_1)$$
$$y_2(L_1 + L_2) = y_3(L_1 + L_2)$$
$$y_1'(L_1) = y_2'(L_1)$$
$$y_2'(L_1 + L_2) = y_3'(L_1 + L_2)$$

Static/Natural

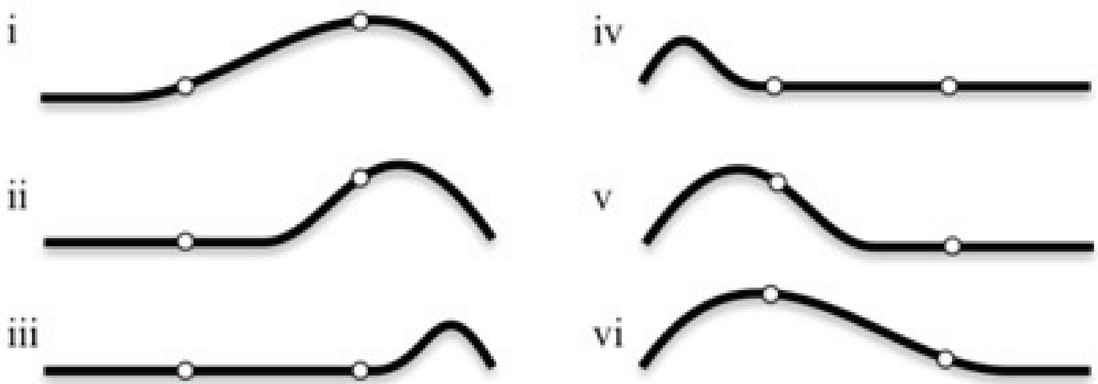
$$y_1''(0) = \kappa_1$$
$$y_3''(L) = \kappa_3$$
$$D_1\left\{y_1''(L_1) - \kappa_1\right\} = D_2\left\{y_2''(L_1) - \kappa_2\right\}$$
$$D_2\left\{y_2''(L_1 + L_2) - \kappa_2\right\} = D_3\left\{y_3''(L_1 + L_2) - \kappa_3\right\}$$
$$D_1 y_1'''\left(L_1\right) = D_2 y_2'''\left(L_2\right)$$
$$D_2 y_2'''\left(L_1 + L_2\right) = D_3 y_3'''\left(L_1 + L_2\right)$$

Step 2: Determine Contact Mode



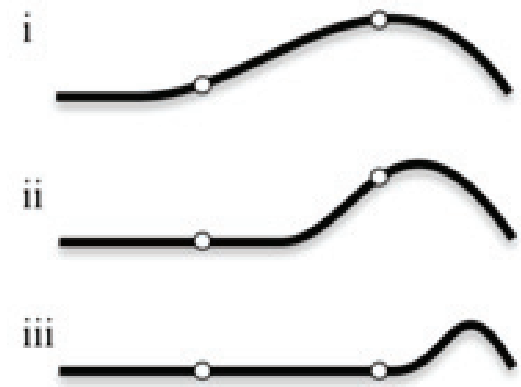
Mode 0: If $y_1'(0) > 0$ and $y_3'(L) < 0$, then robot makes tip contact at its ends (mode 0)

Otherwise, the robot is expected to be engaged in one of the following modes of contact:



Although other contact modes are possible, it is *assumed that the limbs are actuated such that only configurations 0-vi will be observed*.

Mode i-iii: If $y_1'(0) < 0$ and $-y_1'(0) > y_3'(L)$, then left end of robot is expected to engage in "side contact." Additional unknown: length of side contact ξ.



ξ may span 1-3 segments. Since this is not known apriori, we must determine $\{a_i, b_i, c_i, d_i\}$ ξ for each mode:

$$\boxed{\begin{array}{l} 0 < \xi < L_1 \Rightarrow \text{mode i} \\ L_1 < \xi < L_1 + L_2 \Rightarrow \text{mode ii} \\ L_1 + L_2 < \xi < L \Rightarrow \text{mode iii} \end{array}}$$

(i) Replace $y_1(0) = 0 \quad y_1''(0) = \kappa_1$

with $y_1(\xi) = 0 \quad y_1'(\xi) = 0 \quad y_1''(\xi) = 0$

(ii) $y_2(\xi) = y_3(L) = y_2'(\xi) = y_2''(\xi) = 0 \quad y_3''(L) = \kappa_3$

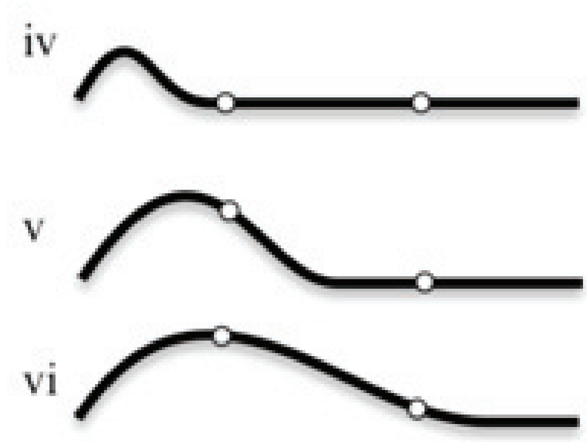
$$y_2(L_1 + L_2) = y_3(L_1 + L_2) \qquad D_2\left\{y_2''(L_1 + L_2) - \kappa_2\right\} = D_3\left\{y_3''(L_1 + L_2) - \kappa_3\right\}$$

$$y_2'(L_1 + L_2) = y_3'(L_1 + L_2) \qquad D_2 y_2'''\left(L_1 + L_2\right) = D_3 y_3'''\left(L_1 + L_2\right)$$

(iii) $y_3(\xi) = y_3(L) = y_3'(\xi) = y_3''(\xi) = 0$

$y_3''(L) = \kappa_3$

Mode iv-vi: If $y_3'(L) > 0$ and $-y_1'(0) < y_3'(L)$, then right end of robot is expected to bein "side contact."



(iv) $y_1(0) = y_1(\xi) = y_1'(\xi) = y_1''(\xi) = 0$

$y_1''(0) = \kappa_1$

(v) $y_1(0) = y_2(\xi) = y_2'(\xi) = y_2''(\xi) = 0$

$y_1(L_1) = y_2(L_2)$

$y_1''(\xi) = \kappa_1$

$D_1\left\{y_1''(L_1) - \kappa_1\right\} = D_2\left\{y_2''(L_1) - \kappa_2\right\}$

$D_1 y_1'''(L_1) = D_2 y_2'''(L_1)$

(vi) Replace $y_3(L) = 0 \quad y_3''(L) = \kappa_1$

with $y_3(\xi) = 0 \quad y_3'(\xi) = 0 \quad y_1''(\xi) = 0$

$0 < \xi < L_1 \Rightarrow$ mode iv
$L_1 < \xi < L_1 + L_2 \Rightarrow$ mode v
$L_1 + L_2 < \xi < L \Rightarrow$ mode vi

Step 3: Determine step length. Vertical deflection y(s) of inextensible rods results in a change in horizontal separation Λ:

$$\Lambda = \int_0^{L_1} \left\{1 - \frac{1}{2}(y_1')^2 \right\} ds + \int_{L_1}^{L_1+L_2} \left\{1 - \frac{1}{2}(y_2')^2 \right\} ds + \int_{L_1+L_2}^{L} \left\{1 - \frac{1}{2}\left(y_3'\right)^2\right\} ds.$$

Step 4: Determine direction of motion. Assume Coulombic friction:

V_t = sliding resistance of tip contact
τa = resistance of side contact

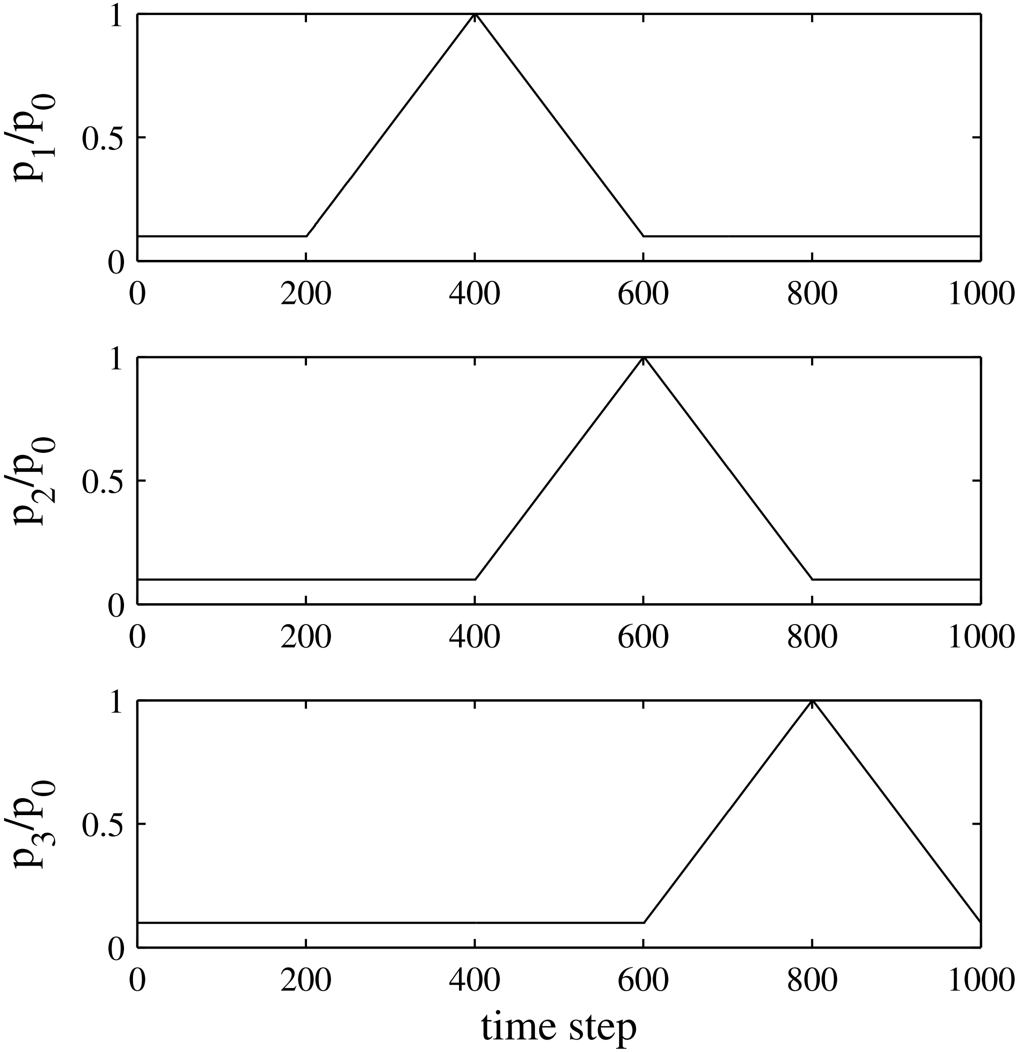
- τ = interfacial shear strength
- a = length of side contact

End with larger sliding resistance remains fixed and the opposite slides to accommodate change in Λ.

Examples

Felt	non-slippery sliding resistance scales with contact Normalized resistance: $V_t = 0, \boldsymbol{\tau} = 1$
Gelatin	slippery deformable sharp tip from point contact digs into substrate flat (side) contact slides Normalized resistance: $V_t = 1, \boldsymbol{\tau} = 0$
Polyethylene	mixed frictional resistance Normalized resistance: $V_t = 0.75, \boldsymbol{\tau} = 0.25$

Input



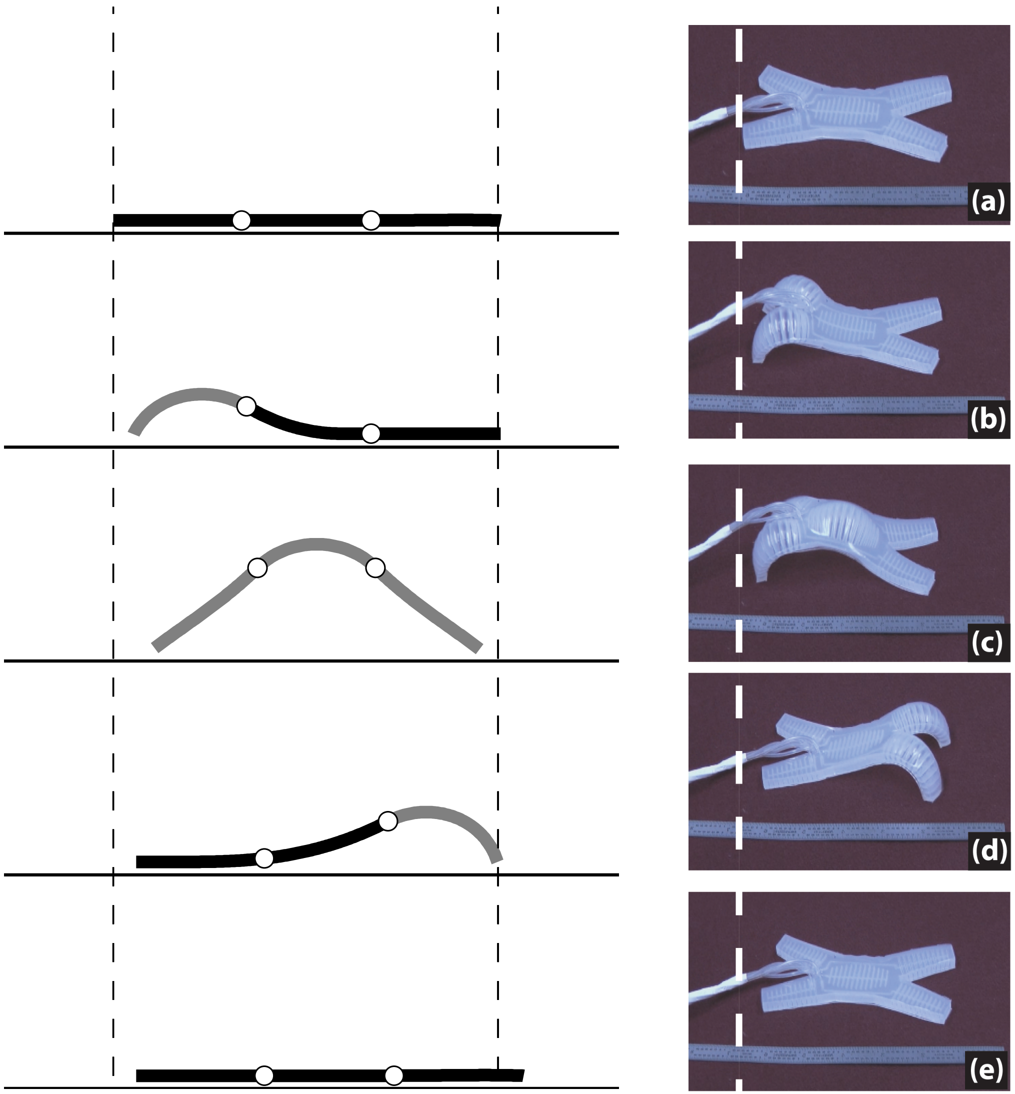
$\mathbf{\kappa_i = \alpha p_i}$

$\mathbf{D_i = D_0 + \beta p_i}$

Unitless analysis
p_i, α, D_0, β are
"normalized"

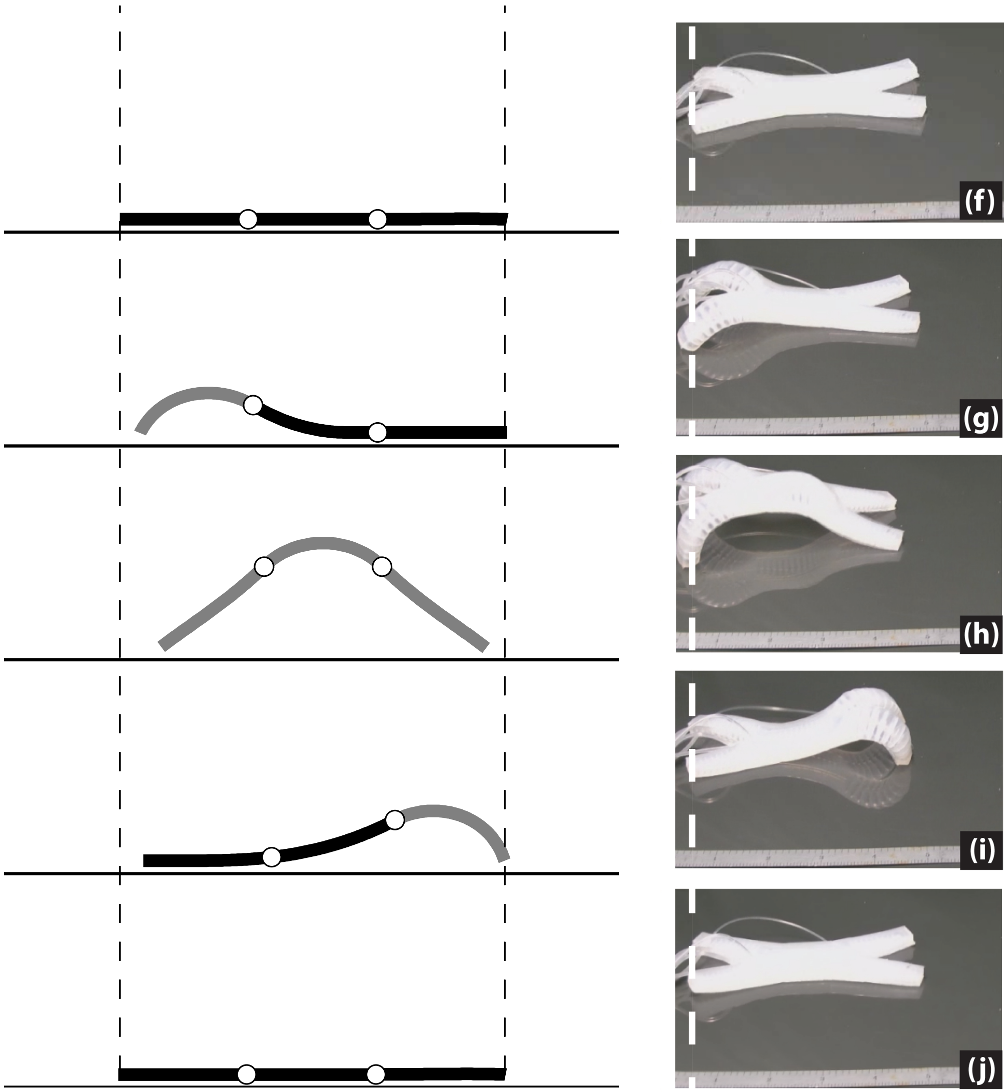
Felt

$V_t = 0, \tau = 1$



Polyethylene

$V_t = 0.75, \tau = 0.25$



Results

Gelatin

$V_t = 1, \tau = 0$