• **Composition of Robot** – Materials & constitutive properties
  - Elastomers, fluids, gas, rigid elements
  - Hyperelasticity – coefficients of elasticity, Poisson’s ratio
  - Dielectric, ferroelectric, or piezoelectric properties (e.g. electric permittivity)
  - Ferromagnetic properties
  - Shape memory or thermal properties (e.g. coefficient of thermal expansion)

• **State of Robot** – Material shape and condition
  - Kinematics – shape & velocities
    - *Reference placement* – initial shape at time $t_0$;
      composed of points $X \in B_0$
    - *Current placement* – current shape at time $t$;
      composed of points $x \in B$
    - Displacement: $u = x - X$
  - Internal voltage field, temperature distribution, magnetic state, …
• **Physical Interactions** – External loads and environmental conditions
  o Mechanical – contact forces (unilateral constraints, friction, collisions), fluid pressure, gravity
  o Electrical – applied electrical field or current (e.g. Maxwell stress, magnetic force)
  o Thermal – temperature change, supplied heat

• **Governing Physics** – Balance Laws
  o *Thermodynamics* – 1st & 2nd Laws; Principle of Minimum Potential
  o *Newton-Euler Equations* – linear and angular momentum balance for entire robot as well as each volumetric or surface element
  o *Maxwell Equations* – balance of electric displacement and magnetic field
• **Position/Orientation of Robot – Global Coordinate Systems (COOS)**
  
  - *Lagrangian Description* – coordinates of initial shape \( (t_0) \):
    \[ X = X_1 i + X_2 j + X_3 k \]
    \[ \nabla_L = \left( \frac{\partial}{\partial X_1} \right) i + \left( \frac{\partial}{\partial X_2} \right) j + \left( \frac{\partial}{\partial X_3} \right) k \]
  
  - *Eulerian Description* – coordinates of current shape \( (t) \):
    \[ x = x_1 i + x_2 j + x_3 k \]
    \[ \nabla_E = \left( \frac{\partial}{\partial x_1} \right) i + \left( \frac{\partial}{\partial x_2} \right) j + \left( \frac{\partial}{\partial x_3} \right) k \]

  “Deformation gradient”: \( F = \nabla_L x \)

  - used to calculate the strain energy density: \( W = W(F) \)
  - Relates the gradient operators: \( \nabla_L = \nabla_E F \)
  - Relates final and initial volumes: \( dV = JdV_0 \), where \( J = \det(F) \)
  - \( T^{1PK} = \partial W / \partial F \) is the “1st Piola Kirchhoff Stress tensor”
  - By definition, Cauchy stress \( \sigma = J^{-1} T^{1PK} F^T \)
The deformation gradient also relates the deformation of surface elements and divergence of the stress tensors:

Consider a surface element that deforms from $d\mathbf{A}_0 = n_0 dA_0$ to $d\mathbf{A} = n dA$. According to Nanson's formula, $d\mathbf{A} = J F^{-T} d\mathbf{A}_0$. This implies the following identity:

$$\nabla L \cdot T^{1PK} dA_0 = \sigma \cdot dA$$

Another useful identity is the Piola transformation (related to Piola identity):

$$\left( \nabla_L \cdot T^{1PK} \right) dV_0 = \left( \nabla_E \cdot \sigma \right) dV$$

ref. A. Betram, Elasticity & Plasticity of Large Deformations 2nd Ed. (2008)
\[ \Pi = \int_{B_0} W dV_0 - \int_B \left\{ \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \right\} dV + \int_{\partial B} \left\{ \eta \Phi - \mathbf{t} \cdot \mathbf{x} \right\} dA \]

\( W \) = strain energy density
\( \Phi = \Phi(x_1,x_2,x_3) \) = voltage
\( \mathbf{E} = \nabla_{\mathbf{E}} \Phi \) = electric field
\( \varepsilon \) = electric permittivity (2\textsuperscript{nd} order tensor; could be anisotropic)
\( \mathbf{D} = \varepsilon \cdot \mathbf{E} \) = electric displacement
\( \eta \) = surface charge (i.e. charge q per unit area)
\( \mathbf{t} \) = surface traction (stress applied to surface)

- \( \Pi = \Pi(x,\Phi) \) – find position and voltage field that minimizes potential energy.
- Determine change in \( \Pi \) when \( x \rightarrow x + \delta x \) and \( \Phi \rightarrow \Phi + \delta \Phi \).
- At equilibrium, corresponding change \( \delta \Pi = 0 \).
Note that \( x \rightarrow x + \delta x \) and \( \Phi \rightarrow \Phi + \delta \Phi \) imply that
\[
\nabla_L x \rightarrow \nabla_L x + \nabla_L \delta x \text{ and } \nabla_E \Phi \rightarrow \nabla_E \Phi + \nabla_E \delta \Phi
\]

\[
\delta \Pi = \int_{B_0} \left\{ T^{1PK} : (\delta \nabla_L x) \right\} dV_0 - \int_{B} \left\{ \frac{1}{2} D \cdot (\delta \nabla_E \Phi) \right\} dV + \int_{\partial B} \{ \eta \delta \Phi - t \cdot \delta x \} dA
\]

Product Rule:

\[
\int_{B_0} \nabla_L \left\{ T^{1PK} \delta x \right\} dV_0 = \int_{B_0} \left\{ (\nabla_L \cdot T^{1PK}) \delta x \right\} dV_0 + \int_{B_0} \left\{ T^{1PK} : (\nabla_L \delta x) \right\} dV_0
\]

\[
\int_{B} \nabla_E \left\{ D \delta \Phi \right\} dV = \int_{B} \left\{ (\nabla_E \cdot D) \delta \Phi \right\} dV + \int_{B} \left\{ D \cdot (\nabla_E \delta \Phi) \right\} dV
\]
Divergence Rule: \[ \int_{B_0} \nabla L \{ T^{1PK} \delta x \} dV_0 = \int_0 \left\{ \left( T^{1PK} \cdot n_0 \right) \delta x \right\} dA_0 \]

\[ \int_{B} \nabla \{ D \delta \Phi \} dV = \int_0 \left\{ \left( D \cdot n \right) \delta \Phi \right\} dA \]

\[ \Rightarrow \int_{B} \{ D \cdot (\nabla \delta \Phi) \} dV = \int_0 \left\{ \left( D \cdot n \right) \delta \Phi \right\} dA - \int_{B} \left\{ (\nabla \cdot D) \delta \Phi \right\} dV \]

\[ \Rightarrow \int_{B_0} \left\{ T^{1PK} : (\nabla_L \delta x) \right\} dV_0 = \int_0 \left\{ \left( T^{1PK} \cdot n_0 \right) \delta x \right\} dA_0 - \int_{B_0} \left\{ (\nabla_L \cdot T^{1PK}) \delta x \right\} dV_0 \]

\[ = \int_0 \left\{ \left( \sigma \cdot n \right) \delta x \right\} dA - \int_{B} \left\{ (\nabla_E \cdot \sigma) \delta x \right\} dV \]

Nanson & Piola identities
Substitute these back into the expression for $\delta \Pi$:

$$\delta \Pi = - \int_B \left\{ \left( \nabla_E \cdot \sigma \right) \delta x + \left( \nabla_E \cdot D \right) \delta \Phi \right\} dV$$

$$+ \int_{\partial B} \left\{ \left( D \cdot n + \eta \right) \delta \Phi + \left( \sigma \cdot n - t \right) \cdot \delta x \right\} dA$$

At equilibrium, $\delta \Pi$ must vanish for any arbitrary choice of $\delta x$ and $\delta \Phi$. This implies the following Balance Laws:

$$\nabla_E \cdot \sigma = 0 \quad \text{for} \quad x \in B \quad \quad \sigma \cdot n = t \quad \text{for} \quad x \in \partial B$$

$$\nabla_E \cdot D = 0 \quad \text{for} \quad x \in B \quad \quad D \cdot n = -\eta \quad \text{for} \quad x \in \partial B$$

Therefore, finding the displacements and voltages becomes a matter of solving a system of PDEs ("Strong Form").

The PDEs are either satisfied or not – we can’t evaluate the accuracy of an approximate solution with a single number.
Approximation

To get an approximate solution, it makes more sense to work with the integral for $\delta \Pi$. This gives us a single number that we can use to evaluate the accuracy of our approximation:

Recall:  
$$\delta \Pi = \int_{B_0} \left\{ T^{1PK} : \left( \nabla_L \delta x \right) \right\} dV_0 - \int_B \left\{ \frac{1}{2} D \cdot \left( \nabla_E \delta \Phi \right) \right\} dV + \int_{\partial B} \left\{ \eta \delta \Phi - t \cdot \delta x \right\} dA$$

The condition $\delta \Pi = 0$ is known as the “Weak Form” of our governing equations:

$$(x, \Phi) \sim$$ unknown functions we need to solve for  
$$(\delta x, \delta \Phi) \sim$$ “weight functions” that are arbitrary

To get an approximate solution, we divide the domain into a finite # of triangular elements.

Between each node, $(x, \Phi, \delta x, \delta \Phi)$ are treated as a linear combination of prescribed basis functions $\phi_i$ (e.g. lines, polynomials/splines).
Finite Element Method

\[ x = \sum_{i=1}^{N} \alpha_i \phi_i(X) \quad \Phi = \sum_{i=1}^{N} \beta_i \phi_i(X) \quad \delta x = \sum_{i=1}^{N} \gamma_i \phi_i(X) \quad \delta \Phi = \sum_{i=1}^{N} \chi_i \phi_i(X) \]

- Here, \((\alpha_i, \beta_i)\) are unknown and \((\gamma_i, \phi_i)\) are arbitrary.
- When we substitute these expressions into \(\delta \Pi\), the integral turns into a summation. Therefore it can be computed by performing matrix operations.
- After some matrix manipulation, the arbitrary values \((\gamma_i, \phi_i)\) drop out. We perform a numerical root finding algorithm to solve for \((\alpha_i, \beta_i)\).
- For nonlinear problems, this is typically done with a gradient descent technique, e.g. Newton-Rapson, Gauss-Newton iteration:
  - Residual is linearized about a certain guess for \((\alpha_i, \beta_i)\)
  - Solving linearized equation leads to a new guess for \((\alpha_i, \beta_i)\)
  - Method will only converge to a local solution if (i) the initial guess is sufficiently close and (ii) linearized matrices are well-conditioned (i.e. non-singular).
- The accuracy of the approximation is evaluated by the convergence of the solution with increasing meshsize \((N)\).

(a) ![Deformation Model](image)

(b) ![Displacement Field](image)

\[ N_a = \text{shape (basis) function} \]

**FEA Discretization:**
\[
\mathbf{x}(\mathbf{X}, t) - \mathbf{X} = \sum N_a(\mathbf{X}) \mathbf{u}_a(t)
\]
\[
\Phi(\mathbf{X}, t) = \sum N_a(\mathbf{X}) \Phi_a(t)
\]

**Weight (test) functions:**
\[
\zeta_i(\mathbf{X}) = \sum N_a(\mathbf{X}) \zeta_{ia}
\]
\[
\eta(\mathbf{X}) = \sum N_a(\mathbf{X}) \eta_a
\]
Free Energy

Arruda-Boyce Model: \[
\frac{W_{\text{stretch}}}{\mu} = \frac{1}{2} (I - 3) + \frac{1}{20N} (I^2 - 9) + \frac{11}{1050N^2} (I^3 - 27) + \frac{19}{7000N^3} (I^4 - 81) + \frac{519}{673750N^4} (I^5 - 243)
\]

Total Free Energy: \[
\hat{W}(\mathbf{C}, \mathbf{E}) = W_{\text{stretch}}(I) + \frac{1}{2} \lambda (\log J)^2 - 2W'_{\text{stretch}}(3) \log J - \frac{\kappa}{2} J C_{ij}^{-1} \tilde{E}_i \tilde{E}_j
\]

1st Piola-Kirchoff Stress: \[
s_{ij} = 2F_{il} \frac{\partial \hat{W}(\mathbf{C}, \mathbf{E})}{\partial C_{jl}}
\]

Electrical Displacement: \[
\tilde{D}_j = -\frac{\partial \hat{W}(\mathbf{C}, \mathbf{E})}{\partial \tilde{E}_j}
\]

\[
\mathbf{C} = \mathbf{F}^T \mathbf{F}
\]

“Right Cauchy-Green Tensor”
Weak Form of PDEs:

**Stress Balance:**
\[
\int s_{ij} \frac{\partial \tilde{\xi}_i}{\partial X_j} dV = \int \left( B_i - \rho \frac{\partial^2 x_i}{\partial t^2} \right) \tilde{\xi}_i dV + \int T_i \tilde{\xi}_i dA
\]

**Maxwell’s Equation:**
\[
- \int \tilde{D}_I \frac{\partial \eta}{\partial X_I} dV = \int q \eta dV + \int \omega \eta dA
\]

**Matrix Form:**
\[
g(u, \Phi, t) = Ma
\]
\[
h(u, \Phi, t) = 0
\]
Matrix Form: \[ g(u, \Phi, t) = M\dot{a} = 0 \text{ (quasistatic)} \]

\[ h(u, \Phi, t) = 0 \]

Linearization:

\[
\begin{bmatrix}
K_{mm} & K_{me} \\
K_{me}^T & K_{ee}
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta \Phi
\end{bmatrix}
= \begin{bmatrix}
f_m \\
f_e
\end{bmatrix}
\]

- \[ K_{mm} = \int H_{ijkl} \frac{\partial N_a}{\partial X_j} \frac{\partial N_b}{\partial X_L} \, dV \]
- \[ K_{me} = -\int e_{ijkl} \frac{\partial N_a}{\partial X_j} \frac{\partial N_b}{\partial X_L} \, dV \]
- \[ K_{ee} = -\int \varepsilon_{ijkl} \frac{\partial N_a}{\partial X_j} \frac{\partial N_b}{\partial X_L} \, dV \]
- \[ f_m = \int_B b_i N_a \, dV + \int_A T_i N_a \, dA - \int q \frac{\partial N_a}{\partial X_j} \, dV \]
- \[ f_e = \int q N_a \, dV + \int \omega N_a \, dA + \int D_j \frac{\partial N_a}{\partial X_j} \, dV \]

- Find roots using Newton-Raphson method (i.e. solve linearized equation at each iteration)
- Instability when Hessian becomes singular
Results

- Fig. 6. Deformation of a dielectric elastomer subject to applied charge loading for various stages of deformation leading to failure of the dielectric elastomer film.
- Fig. 7. Results as shown in Figs. 1–3 were obtained using static and dynamic FEM formulations.
- Fig. 8. The results as shown in Figs. 1–3 were obtained using static and dynamic FEM formulations.
- Fig. 9. The end-to-end distance of each polymer chain approaches a limiting stretch. On approaching the limiting stretch, the elastomer stiffens steeply. This effect is absent in the neo-Hookean model.
- Fig. 10. The results as shown in Figs. 1–3 were obtained using static and dynamic FEM formulations.

In an elastomer, each individual polymer chain has a finite conformation that occurs at large values of applied charge. Because the dynamic electromechanical instability corresponds to a softening in the behavior and electromechanical instability is captured for a limiting stretch. On approaching the limiting stretch, the elastomer stiffens steeply. This effect is absent in the neo-Hookean model:

\[ \varepsilon \approx \frac{1}{C_0} \left( \frac{a}{b-d} \right) \]

where \( \varepsilon \) is the strain, \( C_0 \) is the initial modulus, \( a \) and \( b \) are the equilibrium lengths, and \( d \) is the applied charge. The equa-

We embed the above model into the Sandia-developed simulation code.

5. Numerical results

- Fig. 5. Deformation of a dielectric elastomer subject to applied charge loading shown in Fig. 4.
Results

(a)

(b)

Dielectric Elastomer Quasi-3D Strip

Normalized voltage vs. charge curve for the quasi-3D wrinkling example shown in Fig. 7. The deformation of the elastomer under potential loading is significantly different than that under charge loading in Fig. 4. Due to this enormous increase in film area and thus biaxial stretch behavior of rubber elastic models, the film is unstable under further electrostatic loading. The capability of the proposed dynamic formulation in capturing the electromechanical instabilities in 3D dielectric elastomers is clear that for all values of $y$. The dynamic FEM formulation can be verified by analyzing the voltage vs. charge curve for the 3D strip while the electrostatic loading was applied through an applied voltage was applied to the bottom ($=0$).
CAD MODELING – ISSUES

No algebraic solution
• Scaling laws and design rules?
• How will changing a material property or geometric dimension alter performance?

Takes too long
• Computationally intensive
• Requires hours to run a simulation
• Simulation must be repeated for even minor design changes
• Solution doesn’t always converge

Difficult to validate
• No method to independently validate solution
• Dependent on accuracy of assumptions/input
• “Garbage in, garbage out”
PROPOSED ALTERNATIVE

Discretize!
• Treat each limb as an individual element

Use “Reduce Dimensional” Models
• Euler-Bernoulli Beam Theory
• Kirchoff Plate Theory
• Coulomb’s Friction Law
• Hertzian Contact Theory

Ensure Compatibility
• Attached elements must be *kinematically compatible*
• Transfer of equal-and-opposite loads

Use ODE solvers in MATLAB
• Runga-Kutta (ode45)
• Finite difference (bvp4c)
• Avoid PDEs whenever possible!
EULER-BERNOULLI BEAM THEORY

Deflection are determined by calculating the internal bending moment $M = M(x)$.

\[
\begin{align*}
    v &= \text{deflection} \\
    \theta &= \frac{dv}{dx} = \text{slope} \\
    \kappa &= \frac{d\theta}{dx} = \frac{1}{\rho} = \text{curvature} \\
    \rho &= \text{radius of curvature} \\
    \kappa &= \frac{d\theta}{dx} = \frac{d^2v}{dx^2} \equiv \frac{M}{EI}
\end{align*}
\]
Examples

\[ M = M_0 \]
\[ \frac{d^2v}{dx^2} = \frac{M_0}{EI} \]
\[ v(0) = \frac{dv}{dx}\bigg|_{x=0} = 0 \]

\[ M = V(L - x) \]
\[ \frac{d^2v}{dx^2} = \frac{V}{EI}(L - x) \]
\[ v(0) = \frac{dv}{dx}\bigg|_{x=0} = 0 \]

\[ M = -w(L - x)^2/2 \]
\[ \frac{d^2v}{dx^2} = -\frac{w}{2EI}(L - x)^2 \]
\[ v(0) = \frac{dv}{dx}\bigg|_{x=0} = 0 \]
For large deflections, use Elastica theory

In both Elastica and Linear Beam Theory, $M = D\kappa$, where

$$M = M(\xi) \quad \text{and} \quad \kappa = d\theta/ds$$

The difference between the two theories is how we calculate $M$ and $\theta$.

\[\begin{array}{ll}
\text{Linear Beam Theory} & \text{Elastica} \\
\frac{dv}{ds} = \theta & \frac{dv}{ds} = \sin(\theta) \\
M \text{ calculated in Lagrangian} & M \text{ calculated in Eulerian} \\
\text{Description (Ref. Placement)} & \text{Description (Current)}
\end{array}\]
**Linear Beam Theory**

\[ M = V(L - s) \]

\[ \frac{d\theta}{ds} = \frac{V}{D}(L - s) \]

Converge for small deflection: \( \cos(\theta) \approx 1 \)

\( \frac{dM}{ds} \approx -V \Rightarrow M = C_1 - Vs \)

\( M(L) = 0 \Rightarrow C_1 = Ls \)

\( \Rightarrow \frac{d\theta}{ds} = -(V/D)(L - s) \)

**Elastica**

\[ \frac{dM}{ds} = -V \cos(\theta) \]

\[ \frac{d^2\theta}{ds^2} = -\frac{V}{D} \cos(\theta) \]
In general, \( \frac{d^2 \theta}{ds^2} = k_1 \sin(\theta) + k_2 \cos(\theta) \)

(non-linear 2\textsuperscript{nd}-order ODE)

Must solve for \( \theta(0) = 0 \) and \( \theta'(L) = M_L \)

For pure shear, is determined by solving the following boundary-value problem (BVP):

\[
\frac{d^2 \theta}{ds^2} = -\frac{V}{D} \cos(\theta) \quad \theta(0) = 0 \quad \theta'(L) = 0
\]
solid = Elastica
dashed = Linear Beam Theory \rightarrow \begin{align*}
    x &= s \\
y &= \frac{Vs^2}{6D}(3L - s)
\end{align*}
We typically solve nonlinear BVPs in Matlab using bvp4c:

```matlab
function elastica

global D V R0

n = 100;
L = 1;
D = 1;
V = 2;

s = linspace(0,L,n);
ds = L/(n-1);

solinit = bvpinit(s,@axial_init);
sol = bvp4c(@axial_ode,@axial_bc,solinit);

S = deval(sol,s);
theta = S(1,:);
kappa = S(2,:);
```
\[
\frac{d^2 \theta}{ds^2} = -\frac{V}{D} \cos(\theta) \quad \theta(0) = 0 \quad \theta'(L) = 0
\]

Matlab solves 1\textsuperscript{st} order ODEs (scalars or vectors). Convert:

\[
z = \begin{pmatrix} \theta \\ \theta' \end{pmatrix} \Rightarrow z' = \begin{pmatrix} z_2 \\ -\frac{V}{D} \cos(z_1) \end{pmatrix}
\]

\[
z_1(0) = 0 \quad z_2(L) = 0
\]
solinit = bvpinit(s,@axial_init);
sol = bvp4c(@axial_ode,@axial_bc,solinit);

% ------------------------------------
function dzdr = axial_ode(s,z)

global D V
dzdr = [ z(2); -(V/D)*cos(z(1)) ];

% ------------------------------------
function res = axial_bc(z0,zL)

res = [ z0(1); zL(2) ];

% ------------------------------------
function yinit = axial_init(s)

yinit = [ 0; 0 ];

\[
\frac{d^2\theta}{ds^2} = -\frac{V}{D}\cos(\theta)
\]

\[
\theta(0) = 0 \quad \theta'(L) = 0
\]
\begin{verbatim}
S = deval(sol,s);
theta = S(1,:);
kappa = S(2,:);

x = tril(ones(n,n))*(cos(theta))'*ds;
y = tril(ones(n,n))*(sin(theta))'*ds;

figure(1)
hold on
plot(x,y,'k-')
\end{verbatim}
Deflection: \( v = v(x, y) \)

Estimate deflection using Rayleigh-Ritz method. For a rectangular plate with weight per unit area \( p \):

\[
\Pi = \iint \left\{ \frac{D}{2} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)^2 - vp \right\} \, dx \, dy
\]
Suppose that a rectangular plate with simple supports and edges of length \(a\) and \(b\) is uniformly loaded. To obtain an estimate for \(v = v(x,y)\), we will assume that it has the form

\[
v = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)
\]

This implies

\[
\Pi = \int_0^a \int_0^b \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{D}{2} \left[ a_{mn} \left( \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right]^2 - p_0 a_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right\} \, dx \, dy
\]
Integrating, it follows that

\[
\Pi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\pi^4 abD}{8} a_{mn}^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \frac{4p_0 ab}{\pi^2 mn} a_{mn} \right\}
\]

for \( m,n = 1, 3, \ldots \)

At equilibrium, \( \frac{d\Pi}{da_{mn}} = 0 \)

Solving for \( a_{mn} \) yields

\[
a_{mn} = \frac{16p_0}{\pi^6 mnD \left[ (m/a)^2 + (n/b)^2 \right]^2}
\]
\[ v = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16p_0}{\pi^6 mnD \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right]^2} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \]

for \( m, n = 1, 3, \ldots \)
Review of actuator technologies
Pneumatic • DEA • SMA • IPMC • Bio-hybrid

Representations of Soft Robot Limbs
Euler-Bernoulli Beam • Elastica • Plate/Shell

Simulation of Soft Robot Limbs
Kinematics • Tribology • Analysis (ODE/PDE)
open: \((\kappa_0)_{\text{open}} = 0\)  
closed: \((\kappa_0)_{\text{closed}} = \kappa_0 > 0\)

Deflection: \(v = v(x)\)

- What is \(v(x)\)?
- What is the contact force \(F\)?
- What is the gripping strength?
\[ F \]

\[ y \]

\[ x \]

\[ m = D \left( v'' - \kappa_0 \right) \]

\[ m' = F \Rightarrow v''' = \frac{F}{D} \]

\[ \text{BCs} \]

\[ v(0) = v(L) = v'(0) = 0 \]

\[ m(L) = 0 \Rightarrow v''(L) = \kappa_0 \]
\[ v'' = \frac{F}{D} \Rightarrow v = c_0 + c_1 x + c_2 x^2 + \frac{F}{6D} x^3 \]

\[ v(0) = v'(0) = 0 \Rightarrow v = c_2 x^2 + \frac{F}{6D} x^3 \]

\[ v(L) = 0 \Rightarrow c_2 = -\frac{FL}{6D} \]

\[ v = \frac{F}{6D} \left( x^3 - x^2 L \right) \]

\[ v''(L) = \kappa_0 \Rightarrow F = \frac{3D\kappa_0}{2L} \]

\[ \therefore v = \frac{\kappa_0 x^2}{4} \left( \frac{x}{L} - 1 \right) \]
Coulomb's Law

\[ V = \mu_0 F + \tau A_t \]

**Amonton's Law**

\[ \mu_0 F \sim \text{Mechanical sliding resistance of interlocking asperities} \]

\[ \tau A_t \sim \text{interfacial shear strength} \]

According to Contact Mechanics

\[ A_t \approx A_0 + \alpha F \]

e.g. Greenwood-Williamson & Johnson-Kendall-Roberts Theories
Ignoring the initial adhesion (i.e. $V_0 \approx 0$) slip occurs when

$$2V = \mu_0 F + \tau(\alpha F + A_0) = \mu F + V_0$$

where $\mu = \mu_0 + \tau \alpha$ and $V_0 = \tau A_0$

Limitations of Linearized theory:

- Kinematics not accurate for large deflection
- Does not account for influence of axial load during pick & place operations
Two Finger Gripper

Finger ~ Naturally Curved Elastic rod

- Length $L$, curvature $\kappa_0$
- Natural curvature controlled by actuator
  - open: $(\kappa_0)_{\text{open}}$
  - closed: $(\kappa_0)_{\text{closed}}$
- Flexural rigidity $D = EI$
  - $E = $ Young’s Modulus
  - $I = wh^3/12 = $ Area Moment of Inertia
- Fixed slope at base
- Contact loads $F$ and $V$ at the tip
  - $F = $ normal reaction force to prevent interpenetration
  - $V = $ tangential frictional resistance to sliding
- Large deflection bending
  - Small angle approximation and Euler-Bernoulli beam theory are not valid
  - Use Elastica theory – planar bending; large angle deflection; small bending strains
1 Flexible Bending Actuator. A flexible bending actuator can be treated as an inextensible rod with a natural bending curvature $\kappa_0$ and flexural rigidity $D$. Let $L = 5 \text{ cm}$, $D = 4 \times 10^{-6} \text{ Nm}^2$, $\kappa_0 = 60 \text{ m}^{-1}$.

Suppose that point loads $F$ and $V$ are applied to the free end, as shown. The slope $\theta = \theta(s)$ is determined by minimizing the functional

$$\Pi = \Pi(\kappa) = \int_0^L \frac{1}{2} D_{eq} (\kappa - \kappa_0)^2 \, ds - Fx_L - Vy_L,$$

$$\kappa = d\theta / ds = \theta'$$

$(x_L, y_L)$ are the coordinates of the end of the actuator

**Step 1:** Find $\theta = \theta(s; F, V)$

**Step 2:** Find $x = x(s)$ and $y = y(s)$

**Step 3:** Find $F$ such that $x_L := x(L)$ is equal to $x_0$

**Step 4:** Calculate maximum frictional resistance $\mu F$. If $\mu F > V$ then contact will slip
Step 1: Find $\theta = \theta(s; F, V)$

$$\Pi = \Pi(\kappa) = \int_0^L \frac{1}{2} D_{eq}(\kappa - \kappa_0)^2 \, ds - Fx_L - Vy_L$$

$$x_L = \int_0^L \cos \theta \, ds \quad \text{and} \quad y_L = \int_0^L \sin \theta \, ds$$

Determine $\theta$ that minimizes $\Pi$.

$$\Pi = \int_0^L \left\{ \frac{1}{2} D(\theta' - \kappa_0)^2 - F\cos \theta - V\sin \theta \right\} \, ds = \int_0^L \Gamma(\theta, \theta') \, ds$$

Calculus of Variations:

$$\delta \Pi = \int_0^L \left\{ \frac{\partial \Gamma}{\partial \theta} \delta \theta + \frac{\partial \Gamma}{\partial \theta'} \delta \theta' \right\} \, ds \equiv 0$$

$$= \int_0^L \left\{ \frac{\partial \Gamma}{\partial \theta} \delta \theta + \frac{d}{ds} \left( \frac{\partial \Gamma}{\partial \theta'} \delta \theta \right) - \left[ \frac{d}{ds} \left( \frac{\partial \Gamma}{\partial \theta'} \right) \right] \delta \theta \right\} \, ds \equiv 0$$

$$= \left( \frac{\partial \Gamma}{\partial \theta'} \right)_{s=L} \delta \theta(L) - \left( \frac{\partial \Gamma}{\partial \theta'} \right)_{s=0} \delta \theta(0) + \int_0^L \left\{ \frac{\partial \Gamma}{\partial \theta} - \frac{d}{ds} \left( \frac{\partial \Gamma}{\partial \theta'} \right) \right\} \delta \theta \, ds \equiv 0$$
Step 1: Find $\theta = \theta(s; F, V)$

\[
\theta(0) = 0 \Rightarrow \delta \theta(0) = 0
\]

\[
\Rightarrow \delta \Pi = \left( \frac{\partial \Gamma}{\partial \theta'} \right)_{s=L} \delta \theta(L) + \int_0^L \left\{ \frac{\partial \Gamma}{\partial \theta} - \frac{d}{ds} \left( \frac{\partial \Gamma}{\partial \theta'} \right) \right\} \delta \theta ds
\]

\[
\begin{align*}
\delta \Pi = 0 \forall \delta \theta \Rightarrow \\
\left\{ \\
\left( \frac{\partial \Gamma}{\partial \theta'} \right)_{s=L} = 0 \\
\frac{\partial \Gamma}{\partial \theta} - \frac{d}{ds} \left( \frac{\partial \Gamma}{\partial \theta'} \right) = 0
\right. 
\end{align*}
\]

\[
\frac{\partial \Gamma}{\partial \theta} = F \sin \theta - V \cos \theta
\]

\[
\left( \frac{\partial \Gamma}{\partial \theta'} \right)_{s=L} = 0 \Rightarrow D \{ \theta'(L) - \kappa_0 \} = 0 
\Rightarrow \theta'(L) = \kappa_0
\]

\[
\frac{\partial \Gamma}{\partial \theta} - \frac{d}{ds} \left( \frac{\partial \Gamma}{\partial \theta'} \right) = 0 \Rightarrow F \sin \theta - V \cos \theta - D \theta'' = 0
\]

\[
\therefore \theta'' - \frac{F}{D} \sin \theta + \frac{V}{D} \cos \theta = 0
\]
Boundary Value Problem

\[ \theta'' - \frac{F}{D} \sin \theta + \frac{V}{D} \cos \theta = 0 \quad \theta(0) = 0 \quad \theta'(L) = \kappa_0 \]

In MATLAB, use bvp4c to solve \( \theta \) for \( 0 \leq s \leq L \):

**Step 1a:** Define system parameters & variables – \( F, V, D, \kappa_0, L, s \)

**Step 1b:** Guess Solution

```matlab
solinit = bvpinit(s,@mat4init);
%-------------------------------
function yinit = mat4init(s)
yinit = [ kappa0*s; kappa0 ];
```

**Step 1c:** Define ODE & BCs

let \( z = (\theta \ \theta') \) s.t. \( z_1' = z_2 \) and \( z_2' = (F/D)\sin(z_1) - (V/D)\cos(z_1) \)

```matlab
%-------------------------------
function dzds = mat4ode(s,z)
dzds = [ z(2); (F/D)*sin(z(1)) - (V/D)*cos(z(1)) ];
%-------------------------------
function res = mat4bc(za,zb)
res = [ za(1); zb(2)-kappa0 ];
```
Boundary Value Problem

**Step 1d:** Solve for $z$

```
sol = bvp4c(@mat4ode,@mat4bc,solinit);
```

**Step 1e:** Obtain $\theta$

```
z = deval(sol,s);
theta = z(1,:);
```

**Step 1f:** Plot $\theta$ vs. $s$

```
figure(1); hold on
plot(s*1e3,theta,'k-')
xlabel('s (mm)')
ylabel('\theta (rad)')
```

*Use `global` to pass system parameters (i.e. $F$, $V$, $L$, …) between functions*
Step 2: Find \( x = x(s) \) and \( y = y(s) \)

Given \( \theta = \theta(s) \), we can find \( x \) and \( y \) by integration:

\[
x(s) = \int_0^s \cos \theta(\hat{s}) \, d\hat{s} \quad \text{and} \quad y(s) = \int_0^s \sin \theta(\hat{s}) \, d\hat{s}
\]

Shortcut: Let \( s = \text{linspace}(0, L, n) \) and \( ds = L/(n - 1) \)

\[
\begin{bmatrix}
  s_1 \\
  s_2 \\
  s_3 \\
  \vdots \\
  s_n
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  ds \\
  2ds \\
  \vdots \\
  (n-1)ds = L
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x(s_1) = 0 \\
  x(s_2) \\
  x(s_2) \\
  \vdots \\
  x(s_n)
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  1 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
  \cos \theta(s_1) \\
  \cos \theta(s_2) \\
  \vdots \\
  \cos \theta(s_{n-1})
\end{bmatrix} \, ds
\]

\[
x = [0; \text{tril(ones(n-1,n-1))}*\cos(\text{theta}(1:n-1))'*ds];
\]

\[
y = [0; \text{tril(ones(n-1,n-1))}*\sin(\text{theta}(1:n-1))'*ds];
\]
Step 2: Find $x = x(s)$ and $y = y(s)$
**Step 3:** Find $F$ such that $x_L := x(L)$ is equal to $x_0$

$$x_L = \int_0^L \cos \theta \, ds \equiv x_0$$

In MATLAB, use `fzero`:

```matlab
... 
F_guess = 1e-3;
F = fzero(@get_x0,F_guess);
% -------------------------
function res = get_x0(f)
...
xL = sum(cos(theta)) * ds;
res = xL - x0;
```

**Step 4:** Calculate maximum frictional resistance $\mu F$.
If $\mu F > V$ then contact will slip
3-SEGMENT UNDULATING ROBOT

Felt

Polyethylene

Gelatin
3-Segment Undulating Robot (2D Model)

2 Limb Pairs & 1 Torso
→ 3 Segments

Segments: $S_1, S_2, S_3$
- Lengths $L_i$
- $s = \text{arclength (left to right)}$
- $L = L_1 + L_2 + L_3$
- $S_1 = [0, L_1)$
- $S_2 = [L_1, L_1 + L_2)$
- $S_3 = [L_1 + L_2, L]$
3-Segment Undulating Robot (2D Model)

- Elastic with tunable flexural rigidity $D_i = D(p_i)$ and natural curvature $\kappa_i = \kappa(p_i)$
  - $p_i = \text{signal input (i.e. pressure, voltage, current, ...)}$
  - $y_i = y_i(s)$ vertical deflection
  - $m_i = D(y_i'' - \kappa_i)$ = internal bending moment of $i^{th}$ segment

- Gravitational loading per unit length: $w = (\rho wt)g$
  - $\rho = \text{mass density}$
  - $w = \text{limb width}$
  - $t = \text{limb thickness}$
  - $g = \text{gravity}$

\[
y_i''' = -\frac{w}{D_i} \implies y_i = -\frac{ws^4}{24D_i} + \frac{a_is^3}{6} + \frac{b_is^2}{2} + c_is + d_i
\]
Step 1: At the start of each time step, assume point contact at the ends and calculate the solution \{a_i, b_i, c_i, d_i\} for the following BCs

**Kinematic**

\[ y_1(0) = y_3(L) = 0 \]
\[ y_1(L_1) = y_2(L_1) \]
\[ y_2(L_1 + L_2) = y_3(L_1 + L_2) \]
\[ y_1'(L_1) = y_2'(L_1) \]
\[ y_2'(L_1 + L_2) = y_3'(L_1 + L_2) \]

**Static/Natural**

\[ y_1''(0) = \kappa_1 \]
\[ y_3''(L) = \kappa_3 \]
\[ y_1''(L_1) = y_2''(L_1) \]
\[ D_1 \{y_1''(L_1) - \kappa_1\} = D_2 \{y_2''(L_1) - \kappa_2\} \]
\[ D_2 \{y_2''(L_1 + L_2) - \kappa_2\} = D_3 \{y_3''(L_1 + L_2) - \kappa_3\} \]
\[ D_1 y_1'''(L_1) = D_2 y_2'''(L_2) \]
\[ D_2 y_2'''(L_1 + L_2) = D_3 y_3'''(L_1 + L_2) \]
Step 2: Determine Contact Mode

Mode 0: If \( y_1'(0) > 0 \) and \( y_3'(L) < 0 \), then robot makes tip contact at its ends (mode 0)

Otherwise, the robot is expected to be engaged in one of the following modes of contact:

![Diagram of six contact modes](image)

Although other contact modes are possible, it is assumed that the limbs are actuated such that only configurations 0-vi will be observed.
Mode i-iii: If $y_1'(0) < 0$ and $-y_1'(0) > y_3'(L)$, then left end of robot is expected to engage in “side contact.” Additional unknown: length of side contact $\xi$.

$\xi$ may span 1-3 segments. Since this is not known apriori, we must determine $\{a_i, b_i, c_i, d_i\} \xi$ for each mode:

(i) Replace $y_1(0) = 0 \quad y_1''(0) = \kappa_1$

with $y_1(\xi) = 0 \quad y_1'(\xi) = 0 \quad y_1''(\xi) = 0$

(ii) $y_2(\xi) = y_3(L) = y'_2(\xi) = y''_2(\xi) = 0 \quad y_3''(L) = \kappa_3$

$\quad y_2(L_1 + L_2) = y_3(L_1 + L_2)$

$\quad y'_2(L_1 + L_2) = y'_3(L_1 + L_2)$

$\quad D_2 \{y''_2(L_1 + L_2) - \kappa_2\} = D_3 \{y''_3(L_1 + L_2) - \kappa_3\}$

$\quad D_2 y''_2(L_1 + L_2) = D_3 y''_3(L_1 + L_2)$

(iii) $y_3(\xi) = y_3(L) = y'_3(\xi) = y''_3(\xi) = 0 \quad y''_3(L) = \kappa_3$
Mode iv-vi: If \( y_3'(L) > 0 \) and \( -y_1'(0) < y_3'(L) \), then right end of robot is expected to be in “side contact.”

(iv) \[ y_1(0) = y_1(\xi) = y_1'(\xi) = y_1''(\xi) = 0 \]
\[ y_1''(0) = \kappa_1 \]

(v) \[ y_1(0) = y_2(\xi) = y_2'(\xi) = y_2''(\xi) = 0 \]
\[ y_1(L_1) = y_2(L_2) \]
\[ y_1''(\xi) = \kappa_1 \]
\[ D_1 \{ y_1''(L_1) - \kappa_1 \} = D_2 \{ y_2''(L_1) - \kappa_2 \} \]
\[ D_1 y_1''(L_1) = D_2 y_2''(L_1) \]

(vi) Replace \( y_3(L) = 0 \) \( y_3''(L) = \kappa_1 \)

with \( y_3(\xi) = 0 \) \( y_3'(\xi) = 0 \) \( y_1''(\xi) = 0 \)

\[
\begin{align*}
0 < \xi < L_1 & \Rightarrow \text{mode iv} \\
L_1 < \xi < L_1 + L_2 & \Rightarrow \text{mode v} \\
L_1 + L_2 < \xi < L & \Rightarrow \text{mode vi}
\end{align*}
\]
Step 3: Determine step length. Vertical deflection $y(s)$ of inextensible rods results in a change in horizontal separation $\Lambda$:

$$\Lambda = \int_0^{L_1} \left\{ 1 - \frac{1}{2} (y'_1)^2 \right\} ds + \int_{L_1}^{L_1+L_2} \left\{ 1 - \frac{1}{2} (y'_2)^2 \right\} ds$$

$$+ \int_{L_1+L_2}^L \left\{ 1 - \frac{1}{2} (y'_3)^2 \right\} ds.$$

Step 4: Determine direction of motion. Assume Coulombic friction:

$$V_t = \text{sliding resistance of tip contact}$$
$$\tau a = \text{resistance of side contact}$$

- $\tau = \text{interfacial shear strength}$
- $a = \text{length of side contact}$

End with larger sliding resistance remains fixed and the opposite slides to accommodate change in $\Lambda$. 
### Examples

<table>
<thead>
<tr>
<th>Material</th>
<th>Properties</th>
<th>Normalized Resistance: ( V_t = ), ( \tau = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Felt</td>
<td>non-slippery</td>
<td>( V_t = 0, \tau = 1 )</td>
</tr>
<tr>
<td></td>
<td>sliding resistance scales with contact</td>
<td></td>
</tr>
<tr>
<td>Gelatin</td>
<td>slippery</td>
<td>( V_t = 1, \tau = 0 )</td>
</tr>
<tr>
<td></td>
<td>deformable</td>
<td></td>
</tr>
<tr>
<td></td>
<td>sharp tip from point contact digs into substrate</td>
<td></td>
</tr>
<tr>
<td></td>
<td>flat (side) contact slides</td>
<td></td>
</tr>
<tr>
<td>Polyethylene</td>
<td>mixed frictional resistance</td>
<td>( V_t = 0.75, \tau = 0.25 )</td>
</tr>
</tbody>
</table>
\[ \kappa_i = \alpha p_i \]
\[ D_i = D_0 + \beta p_i \]

Unitless analysis

\( p_i, \alpha, D_0, \beta \) are “normalized”
Results

Felt

$V_t = 0, \tau = 1$
Results

Polyethylene

\[ V_t = 0.75, \tau = 0.25 \]
Results

Gelatin

$V_t = 1, \tau = 0$