“DEMES” Frame-based Flexural DEA

Membrane

Frame

$\lambda_0$

$\Phi = 0 \, kV \quad 2 \, kV$

$3 \, kV \quad 4 \, kV$

Here, $\lambda$ is the frame curvature and $\cos(\phi)$ is the prestretch.

Since the frame is primarily loaded by axial tension from the current placement of the membrane’s midplane w.r.t. the frame, it will be assumed that the membrane is incompressible, the condition $\epsilon_{r} = 0$. If the membrane is treated as an incompressible Neo-Hookean solid with coefficients $\bar{E} = 2 \, MPa$, $\bar{h} = 0.1 \, mm$, $\bar{m} = 8.85 \times 10^{-12} \, F/(m \cdot kV)$, and $\bar{r} = 1 \, \mu e$, the space curve formed by the convecting surface coordinates $E_t$ is tension is sustained.

The border of the frame has a thick-line (for the space curve formed by the convecting surface coordinates $E_t$ is tension is sustained.)

Let $\lambda = \pi \times \cos(\phi)$ and $\kappa \eta = \chi (\ell \times s)$, where $\phi$ is the principle stretch along the $y$-direction, and Young’s modulus $E/h$ is computed as $\epsilon_{s} = \pi \times \cos(\phi)$.

The exposed surfaces of the membrane are coated with electrodes that remain intact (and electrically insulated) from the dielectric is treated as an incompressible Neo-Hookean solid with coefficients $\bar{E} = 2 \, MPa$, $\bar{h} = 0.1 \, mm$, $\bar{m} = 8.85 \times 10^{-12} \, F/(m \cdot kV)$, and $\bar{r} = 1 \, \mu e$. This is admissible since the membrane is naturally flat in the principal direction, and Young’s modulus $E/h$ is computed as $\epsilon_{s} = \pi \times \cos(\phi)$.
Theoretical Model

Curvilinear Coord. \((s, y)\) convects with the plane of the bending frame

\[
\begin{align*}
\Phi = 0 \text{ kV} & \quad 2 \text{ kV} \\
3 \text{ kV} & \quad 4 \text{ kV}
\end{align*}
\]
Theoretical Model

Curvilinear Coord. \((s, y)\) convects with the plane of the bending frame.

Frame Deflection:

\[
e_s = e_x \cos \phi + e_y \sin \phi
\]

\[
e_n = -e_x \sin \phi + e_y \cos \phi
\]

\[
\kappa = \frac{d\phi}{ds}
\]

Membrane Deformation:

\[
u = u(s, y) = u_s e_s + u_y e_y + \eta e_n
\]

“Dirilichet” Boundary Conditions:

\[
u_s = u_y = \eta = 0 \quad \forall X \in \partial S
\]
Modeling is challenging because we must simultaneously examine the deformation of both the frame and the membrane.

Use Principle of Minimum Potential:

\[ \Pi = \int_{\Omega_0} \Psi \, dv + \int_{\Omega} \Gamma \, dv + \int_{\mathcal{F}} U \, dA \]

\( \lambda_s = \ell / L_0 \quad \lambda_y = w / W_0 \)

Frame will bend if prestretch \( \lambda_0 \) is greater than critical buckling load.
Theoretical Model

\[ \Pi = \int_{\Omega_0} \Psi \, dV + \int_{\Omega} \Gamma \, dv + \int_{\mathcal{F}} U \, dA \]

Elastic strain energy density: \( \Psi = C_1 \left\{ \lambda_s^2 \left[ (\eta, s + \kappa u_s)^2 + (1 + u_s, s - \kappa \eta)^2 + u_{y, s}^2 \right] \right\} + \lambda_y^2 \left[ (1 + u_y, y)^2 + \eta_{, y}^2 + u_{s, y}^2 \right] + \frac{1}{\lambda_s^2 \lambda_y^2 \left[ \eta_{, s}^2 + (1 - \kappa \eta)^2 (1 + \eta_{, y}^2) \right]} - 3 \)

Bending energy of frame: \( \int_{-\ell / 2}^{\ell / 2} \frac{1}{2} D \kappa^2 \, ds \)

Electrical enthalpy: \( \Gamma = -\frac{1}{2} \epsilon \left( \frac{\Phi}{\mu H} \right)^2 \)

Stretch through thickness: \( \mu \approx \frac{1}{\lambda_s \lambda_y \sqrt{\eta_{, s}^2 + (1 - \kappa \eta)^2 (1 + \eta_{, y}^2)}} \)
Theoretical Model

\[ \Pi = \int_{\Omega_0} \Psi \, dV + \int_{\Omega} \Gamma \, dv + \int_{\mathcal{F}} U \, dA \]

\[ = \int_{-\ell/2}^{\ell/2} \int_{-w/2}^{w/2} \mathcal{L} \, dyds + \int_{-\ell/2}^{\ell/2} \frac{1}{2} D\kappa^2 \, ds \]

“Lagrangian density”: \( \mathcal{L} = (\Psi + \Gamma) H/\lambda_s \lambda_y \)

A condition for \( \Pi \) to be minimized is that each function \( \varphi(s, y) \in \{\eta, u_s, u_y\} \) Must minimize the “Euler-Lagrange equations”:

\[ \mathcal{L}_\varphi - (\partial \mathcal{L}/\partial \varphi)_s, s - (\partial \mathcal{L}/\partial \varphi)_y, y = 0 \quad \varphi = 0 \quad \forall \mathbf{x} \in \partial S \]

The proof of this is obtained using the “Calculus of Variations”.

These PDEs are equivalent to the Newtonian equations for balancing internal stresses in the membrane (in the frame and membrane).
Theoretical Model

This implies \( u_y = 0 \) and that \( \eta \) and \( u_s \) must satisfy the following system of “elliptic” PDEs:

\[-\nabla \cdot (c_1 \nabla \eta) + a_1 \eta = f_1\]
\[-\nabla \cdot (c_2 \nabla u_s) + a_2 u_s = f_2\]

\[
c_1 = \left\{ \lambda_s^2 \left( 1 - \lambda_y^2 \mu^4 \right) - \frac{\lambda_s \lambda_y \epsilon \Phi^2}{2C_1 H} \right\} e_s \otimes e_s + \left\{ \lambda_y^2 \left( 1 - \lambda_s^2 \mu^4 (1 - \kappa \eta)^2 \right) - \frac{\lambda_s \lambda_y \epsilon \Phi^2}{2C_1 H} \right\} e_y \otimes e_y
\]

\[
c_2 = \lambda_s^2 e_s \otimes e_s + \lambda_y^2 e_y \otimes e_y
\]

\[
a_1 = \lambda_s^2 \kappa^2 \left\{ 1 - \lambda_y^2 \mu^4 (1 + \eta, y)^2 \right\} - \frac{\lambda_s \lambda_y \epsilon \Phi^2}{2C_1 H} \kappa^2 (1 + \eta^2)
\]

\[
a_2 = \lambda_s^2 \kappa^2
\]

\[
f_1 = \lambda_s^2 \kappa \left\{ 1 + 2u_{s,s} - \lambda_y^2 \mu^4 (1 + \eta, y)^2 \right\} + \lambda_s^2 \kappa, s u_s - \frac{\lambda_s \lambda_y \epsilon \Phi^2}{2C_1 H} \kappa (1 + \eta, y)
\]

\[
f_2 = -\lambda_s^2 (\kappa, s \eta + 2 \kappa \eta, s)
\]
Another condition for static equilibrium is that $\Pi$ must be minimized w.r.t. $\kappa$.

\[
\frac{\partial \Pi}{\partial \kappa} = 0 \Rightarrow \int_{-w/2}^{w/2} \frac{\partial L}{\partial \kappa} \, dy + D\kappa = 0
\]

Simultaneously solving this along with the PDEs requires FEA (e.g. ANSYS, ABAQUS, COMSOL).

To simplify the analysis, we note that the frame is primarily loaded by axial tension from the membrane. From the post-buckling solution of an Euler column we assume that $\kappa$ may be approximated as

\[
\kappa = \frac{d\phi}{ds} \approx \alpha_0 \cos \left( \frac{\pi s}{\ell} \right)
\]
Solution Method

1. Select a voltage $V$

2. Determine $\alpha$
   2.1 Use `fminbnd` to find $\alpha$ that minimizes $\Pi$
   2.2 For each value of $\alpha$, calculate $\Pi$
      2.2.1 Use `pdenonlin` to solve PDEs and obtain \{\eta, u_s, u_y\}
      2.2.2 Numerically integrate Lagrangian density
      2.2.3 Calculate bending energy of frame: $U_f = D\ell\alpha^2/4$

3. Calculate total bending angle of frame: $\phi = \alpha\ell/\pi$
function phi = plot_phi

    global ell
    ell = 2e-2;
    n = 20;
    Voltage = linspace(0,20e3,n);

    for i = 1:n
        alpha = get_alpha(Voltage(i));
        phi(i) = (alpha*ell/pi)*180/pi;
    end

    plot(Voltage*1e-3,phi,'k-',Voltage*1e-3,-phi,'k-')
    xlabel('\Phi (kV)')
    ylabel('\phi (degrees)')

function alpha = get_alpha(V)

    global Phi u0

    u0 = 0;
    Phi = V;

    alpha = fminbnd(@get_Pi,0,200)
function Pi = get_Pi(a);

    global Phi u0 ell

    model = createpde(2);

    E = 170e3;
    C1 = E/6;
    H = 130e-6;
    eps = 2*8.85e-12;
    ls = 2;
    ly = 1;
    w = 1e-2;

    Ef = 1e6;
    wf = 1e-2;
    hf = 1.3e-3;
    D = Ef*wf*hf^3/12;
    alpha = a;

    R1 = [3,4,-ell/2,ell/2,ell/2,-ell/2,-w/2,-w/2,w/2,w/2]';
    gm = [R1];
    ns = char('R1');
    sf = 'R1';
    g = decsg(gm,sf,ns);
    geometryFromEdges(model,g);
\[-\nabla \cdot (\mathbf{c}_1 \nabla \eta) + a_1 \eta = f_1\]
\[-\nabla \cdot (\mathbf{c}_2 \nabla u_s) + a_2 u_s = f_2\]

```matlab
k = sprintf('d*cos(pi*x/d)', alpha, ell);
k = sprintf('-d*pi*sin(pi*x/d)', alpha, ell, ell);
mu = sprintf('1/(d*d*sqrt(ux(1,:).^2 + (1 - s.*u(1,:)).^2.*((1 + uy(1,:).^2))'))', ...
ls, ly, k);
gamma = sprintf('d*d*d*d*(d.^2)/(2*d*d)', ls, ly, eps, Phi, C1, H);

c1 = sprintf('d^2.*(1 - (d.^2).*s.^4) - s', ls, ly, mu, gamma);
c2 = sprintf('d^2.*(1 - (d.^2).*s.^4.*((1 - s.*u(1,:)).^2)) - s.*(1 - s.*u(1,:)).^2)', ...
ly, ls, mu, k, gamma, k);
c3 = sprintf('d^2', ls);
c4 = sprintf('d^2', ly);

a1 = sprintf('(d^2)*s.^2.*(1 - (d.^2).*s.^4.*((1 + uy(1,:)).^2)) - s.*(s.^2)', ...
*(1 + uy(1,:).^2), ls, k, ly, mu, gamma, k);
a2 = sprintf('(d^2)*s.^2', ls, k);

f1 = sprintf('(d^2)*s.*(1 + 2*ux(2,:)) - (d.^2).*s.^4.*((1 + uy(1,:)).^2) + s.*u(2,:)) - s.*(s.^2).*((1 + uy(1,:).^2)', ...
ls, k, ly, mu, ks, gamma, k);
f2 = sprintf('-d^2*s.*u(1,:) + 2*s.*ux(1,:)', ls, ks, k);

c = char(c1, '0', '0', c2, c3, '0', '0', c4);
a = char(a1, a1);
f = char(f1, f2);
```
```matlab
applyBoundaryCondition(model,'Edge',1:model.Geometry.NumEdges,...
    'u',[0,0],'Vectorized','on');
mesh = generateMesh(model);

u = pdenonlin(model,c,a,f,'Tol',1e-8,'U0',u0);

u0 = u; % Update initial guess
numNodes = size(model.Mesh.Nodes,2);

n = 100;
s = linspace(-ell/2,ell/2,n);
ds = ell/(n-1);
y = linspace(-w/2,w/2,n);
dy = w/(n-1);
kappa = alpha*cos(pi*s/ell);

[p,e,t] = meshToPet(mesh);
[ux,uy] = pdegrad(p,t,u);

eta = tri2grid(p,t,u,s,y);
eta_s = tri2grid(p,t,ux(1,:)',s,y);
eta_y = tri2grid(p,t,uy(1,:)',s,y);
us = tri2grid(p,t,u((numNodes+1):2*numNodes),s,y);
us_s = tri2grid(p,t,ux(2,:)',s,y);
us_y = tri2grid(p,t,uy(2,:)',s,y);
```
Results

\( \ell = 2 \text{ cm}, \ w = 1 \text{ cm}, \ H = 130 \ \mu\text{m}, \)
\( E = 170 \text{ kPa}, \ C_1 = E/6, \ \epsilon = \epsilon_r \epsilon_0, \ \epsilon_r = 2, \)
\( E_f = 1 \text{ MPa}, \ w_f = 1 \text{ cm}, \ h_f = 1.3 \text{ mm}, \)

\( \Phi = 0, \ \lambda_s = 2, \text{ and } \lambda_y = 1 \)
\( \ell = 2 \text{ cm}, \ w = 1 \text{ cm}, \ H = 130 \ \mu \text{m}, \)
\( E = 170 \ \text{kPa}, \ C_1 = E/6, \ \epsilon = \epsilon_r \epsilon_0, \ \epsilon_r = 2, \)
\( E_f = 1 \ \text{MPa}, \ w_f = 1 \ \text{cm}, \ h_f = 1.3 \ \text{mm}, \)

\( \lambda_s = 2, \ \text{varying} \ \Phi \)

\( \psi \) (degrees)

\( \Phi \) (kV)

\( \psi \) (degrees)

\( \lambda_s \) (degrees)

\( \Phi = 0 \)

\( \Phi = 5 \ \text{kV} \)
Simplified DEMES Model

Model and design of dielectric elastomer minimum energy structures


Assumptions:

- Membrane detached along side
- Frame (modulus = $Y_f$) bends into a pure circle
- Membrane = Gent solid with shear modulus $\mu$.

$$U_{tot}(\theta, V) = \frac{1}{2} \frac{Y_f \cdot b \cdot d^3}{12 \cdot c} \theta^2 - \frac{c \cdot w \cdot t_0}{\lambda_p} \frac{\mu J_m}{2}$$

$$\times \ln \left( 1 - \frac{\lambda^2 + \lambda^{-2} - 2}{J_m} \right) - \frac{e \cdot c \cdot w \cdot V^2 \lambda^2}{2 t_0 \lambda_p}$$
Simplified DEMES Model

\[ U_{\text{tot}}(\theta, V) = \frac{1}{2} \frac{Y_f \cdot b \cdot d^3}{12 \cdot c} \theta^2 - \frac{c \cdot w \cdot t_0}{\lambda_p} \frac{\mu J_m}{2} \ln \left( 1 - \frac{\lambda^2 + \lambda^{-2} - 2}{J_m} \right) - \frac{\epsilon \cdot c \cdot w \cdot V^2 \lambda^2}{2t_0 \lambda_p} \]

\[ \frac{\partial U_{\text{tot}}}{\partial \theta} = 0 \]

No Voltage
Influence of membrane thickness and prestretch
Simplified DEMES Model

\[ U_{tot}(\theta, V) = \frac{1}{2} \frac{Y_f \cdot b \cdot d^3}{12 \cdot c} \theta^2 - \frac{c \cdot w \cdot t_0}{\lambda_p} \mu J_m \frac{\mu J_m}{2} \times \ln \left( 1 - \frac{\lambda^2 + \lambda^{-2} - 2}{J_m} \right) - \frac{\epsilon \cdot c \cdot w \cdot V^2 \lambda^2}{2 t_0 \lambda_p} \]

\[ \Theta_{min} \]

\[ \Theta_0 \]

\[ V=0 \]

\[ V=V_{max} \]

\[ E_b = 70 \text{ kV/mm} \]
Creep

Figure 10. PET frames rolled around a post and left for one night at 80° (left) exhibited a significant creep when released from the post (right).