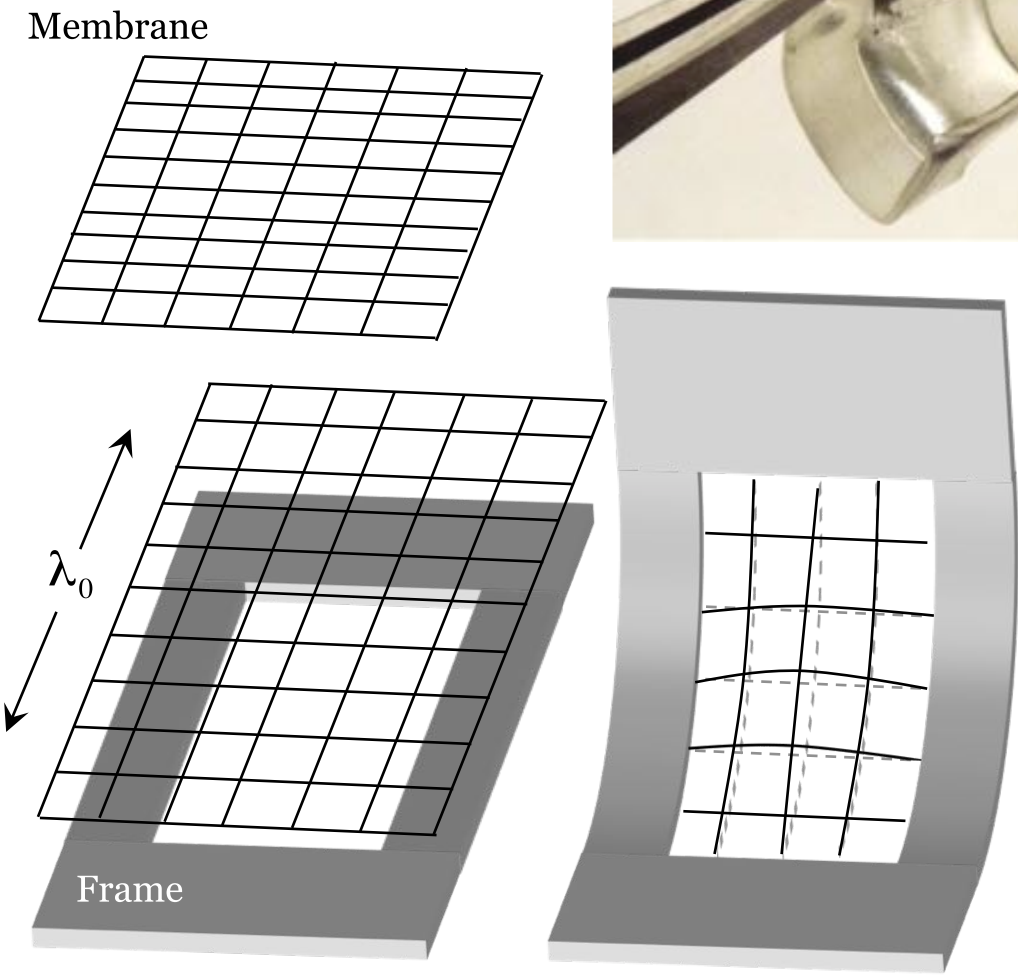
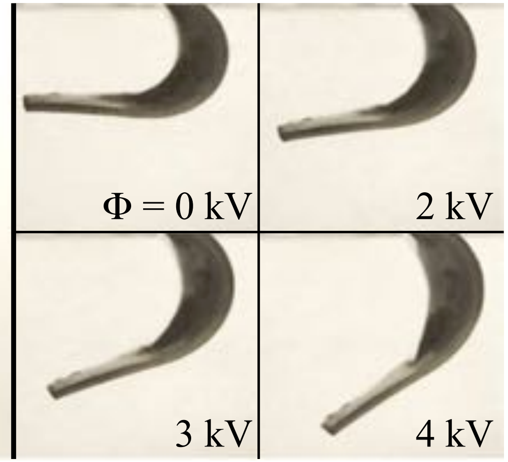
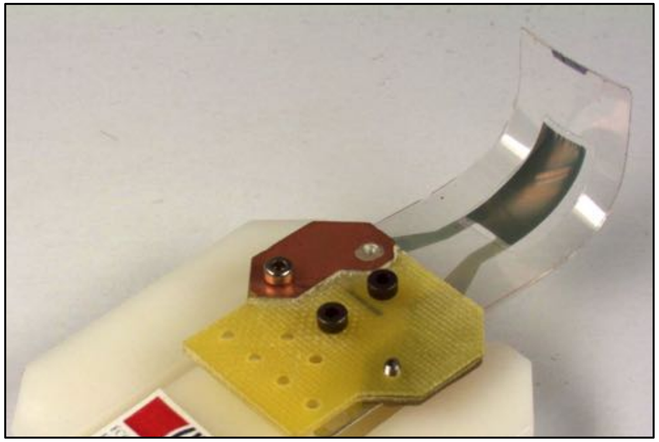
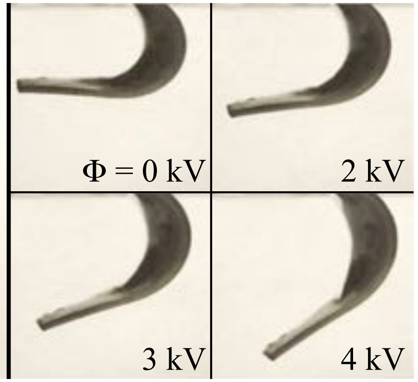
"DEMES" Frame-based Flexural DEA

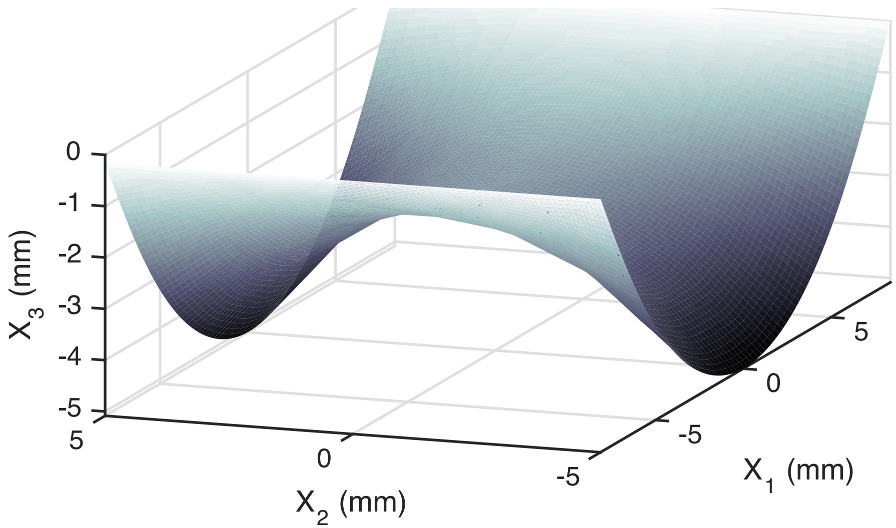


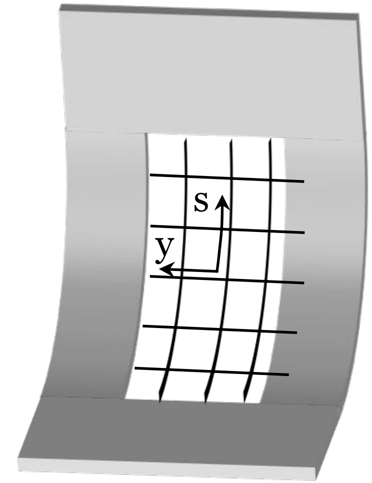




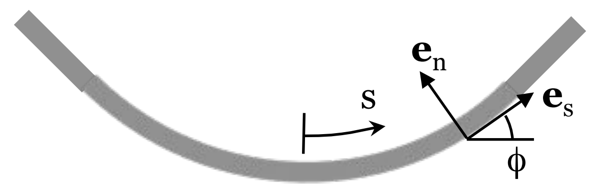
Theoretical Model





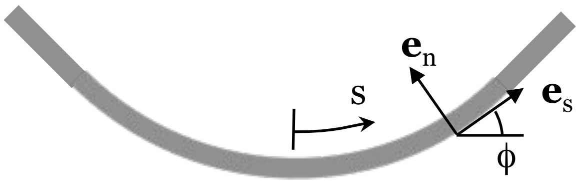


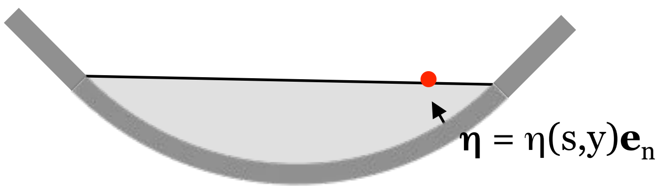
Curvilinear Coord. (s, y) convects with the plane of the bending frame



Theoretical Model

Curvilinear Coord. (s, y) convects with the plane of the bending frame





Frame Deflection:

$$\mathbf{e}_s = \mathbf{e}_x \cos\phi + \mathbf{e}_y \sin\phi$$

$$\mathbf{e}_n = -\mathbf{e}_x \sin\phi + \mathbf{e}_y \cos\phi$$

$$\boldsymbol{\kappa} = \frac{d\boldsymbol{\phi}}{ds}$$

Membrane Deformation:

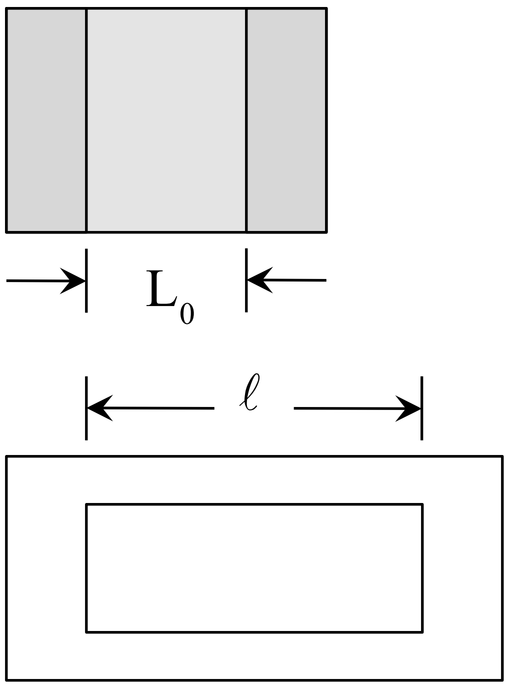
$$\mathbf{u} = \mathbf{u}(s, y) = u_s \mathbf{e}_s + u_y \mathbf{e}_y + \eta \mathbf{e}_n$$

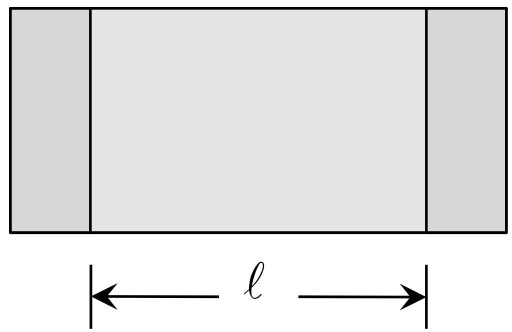
“Dirilichet” Boundary Conditions:

$$u_s = u_y = \eta = 0 \quad \forall \mathbf{X} \in \partial S$$

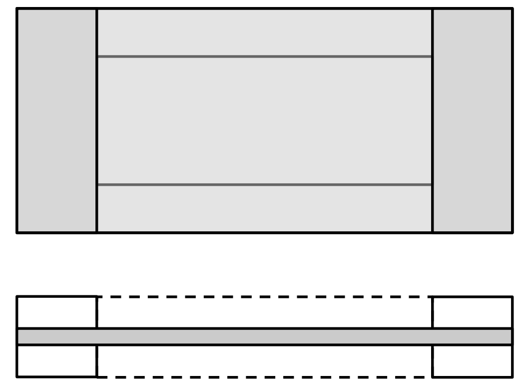
Modeling is challenging because we must simultaneously examine the deformation of both the frame and the membrane.

Use Principle of Minimum Potential: $\Pi = \int_{\Omega_0} \Psi \, dV + \int_{\Omega} \Gamma \, dv + \int_{\mathscr{F}} U \, dA$





Prestretch Dielectric



$\lambda_s = \ell / L_0 \quad \lambda_y = w / W_0$

Frame will bend if prestretch λ_0 is greater than critical buckling load.

Theoretical Model

$$\Pi = \int_{\Omega_0} \Psi \, dV + \int_{\Omega} \Gamma \, dv + \int_{\mathscr{F}} U \, dA$$

Elastic strain energy density: $\Psi = C_1 \Big\{ {\lambda_s}^2 \left[(\eta_{,s} + \kappa u_s)^2 + (1 + u_{s,s} - \kappa\eta)^2 + u_{y,s}^2 \right] + {\lambda_y}^2 \left[(1 + u_{y,y})^2 + \eta_{,y}^2 + u_{s,y}^2 \right] + \dfrac{1}{{\lambda_s}^2 {\lambda_y}^2 \left[\eta_{,s}^2 + (1-\kappa\eta)^2 (1 + \eta_{,y}^2) \right]} - 3 \Big\}$

Bending energy of frame: $\displaystyle\int_{-\ell/2}^{\ell/2} \frac{1}{2} D\kappa^2 \, ds$

Electrical enthalpy: $\Gamma = -\dfrac{1}{2}\epsilon \left(\dfrac{\Phi}{\mu H} \right)^2$

Stretch through thickness: $\mu \approx \dfrac{1}{\lambda_s \lambda_y \sqrt{\eta_{,s}^2 + (1-\kappa\eta)^2 (1 + \eta_{,y}^2)}}$

Theoretical Model

$$\Pi = \int_{\Omega_0} \Psi \, dV + \int_{\Omega} \Gamma \, dv + \int_{\mathscr{F}} U \, dA$$
$$= \int_{-\ell/2}^{\ell/2} \int_{-w/2}^{w/2} \mathscr{L} \, dy ds + \int_{-\ell/2}^{\ell/2} \frac{1}{2} D\kappa^2 \, ds$$

"Lagrangian density": $\mathscr{L} \;=\; (\Psi + \Gamma)H/\lambda_s\lambda_y$

A condition for Π to be minimized is that each function $\varphi(s, y) \;\in \{\eta, u_s, u_y\}$
Must minimize the "Euler-Lagrange equations":

$$\mathscr{L}_\varphi - (\partial \mathscr{L}/\partial \varphi_{,s})_{,s} - (\partial \mathscr{L}/\partial \varphi_{,y})_{,y} \;=\; 0 \qquad \varphi = 0 \quad \forall \mathbf{X} \in \partial S$$

The proof of this is obtained using the "Calculus of Variations".

These PDEs are equivalent to the Newtonian equations for balancing internal stresses in the membrane (in the frame and membrane).

Theoretical Model

This implies $u_y = 0$ and that η and u_s must satisfy the following system of "elliptic" PDEs:

$$-\nabla\cdot\left(\mathbf{c}_1\nabla\eta\right)+a_1\eta=f_1$$

$$-\nabla\cdot\left(\mathbf{c}_2\nabla u_s\right)+a_2u_s=f_2$$

$$\begin{aligned}\mathbf{c}_1 = &\left\{{\lambda_s}^2\left(1-{\lambda_y}^2\mu^4\right)-\frac{\lambda_s\lambda_y\epsilon\Phi^2}{2C_1H}\right\}\mathbf{e}_s\otimes\mathbf{e}_s\\ &+\left\{{\lambda_y}^2\left(1-{\lambda_s}^2\mu^4(1-\kappa\eta)^2\right)\right.\\ &\left.\qquad-\frac{\lambda_s\lambda_y\epsilon\Phi^2}{2C_1H}(1-\kappa\eta)^2\right\}\mathbf{e}_y\otimes\mathbf{e}_y\\ \mathbf{c}_2 = &{\lambda_s}^2\mathbf{e}_s\otimes\mathbf{e}_s+{\lambda_y}^2\mathbf{e}_y\otimes\mathbf{e}_y\end{aligned}$$

$$\begin{aligned}a_1 &= {\lambda_s}^2\kappa^2\left\{1-{\lambda_y}^2\mu^4(1+\eta_{,y})^2\right\}\\ &\quad-\frac{\lambda_s\lambda_y\epsilon\Phi^2}{2C_1H}\kappa^2(1+\eta_{,y}^2)\\ a_2 &= {\lambda_s}^2\kappa^2\end{aligned}$$

$$\begin{aligned}f_1 &= {\lambda_s}^2\kappa\left\{1+2u_{s,s}-{\lambda_y}^2\mu^4(1+\eta_{,y})^2\right\}\\ &\quad+{\lambda_s}^2\kappa_{,s}u_s-\frac{\lambda_s\lambda_y\epsilon\Phi^2}{2C_1H}\kappa(1+\eta_{,y}^2)\\ f_2 &= -{\lambda_s}^2(\kappa_{,s}\eta+2\kappa\eta_{,s})\end{aligned}$$

Approximation

Another condition for static equilibrium is that Π must be minimized w.r.t. κ.

$$\frac{\partial \Pi}{\partial \kappa} = 0 \Rightarrow \int_{-w/2}^{w/2} \frac{\partial L}{\partial \kappa} dy + D\kappa = 0$$

Simultaneously solving this along with the PDEs requires FEA (e.g. ANSYS, ABAQUS, COMSOL).

To simplify the analysis, we note that the frame is primarily loaded by axial tension from the membrane. From the post-buckling solution of an Euler column we assume that κ may be approximated as

$$\kappa = \frac{d\phi}{ds} \approx \alpha_0 \cos\left(\frac{\pi s}{\ell}\right)$$

Solution Method

1. Select a voltage V

2. Determine α
    - 2.1 Use 'fminbnd' to find α that minimizes Π
    - 2.2 For each value of α, calculate Π
        - 2.2.1 Use 'pdenonlin' to solve PDEs and obtain $\{\eta, u_s, u_y\}$
        - 2.2.2 Numerically integrate Lagrangian density
        - 2.2.3 Calculate bending energy of frame: $Uf = D\ell\alpha^2/4$

3. Calculate total bending angle of frame: $\phi = \alpha\ell/\pi$

Code

```
function phi = plot_phi

    global ell

    ell = 2e-2;

    n = 20;
    Voltage = linspace(0,20e3,n);

    for i = 1:n
        alpha = get_alpha(Voltage(i));
        phi(i) = (alpha*ell/pi)*180/pi;
    end

    plot(Voltage*1e-3,phi,'k-',Voltage*1e-3,-phi,'k-')
    xlabel('\Phi (kV)')
    ylabel('\phi (degrees)')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function alpha = get_alpha(V)

    global Phi u0

    u0 = 0;
    Phi = V;

    alpha = fminbnd(@get_Pi,0,200)
```

Code

```
function Pi = get_Pi(a);

    global Phi u0 ell

    model = createpde(2);

    E = 170e3;
    C1 = E/6;
    H = 130e-6;
    eps = 2*8.85e-12;
    ls = 2;
    ly = 1;
    w = 1e-2;

    Ef = 1e6;
    wf = 1e-2;
    hf = 1.3e-3;
    D = Ef*wf*hf^3/12;
    alpha = a;

    R1 = [3,4,-ell/2,ell/2,ell/2,-ell/2,-w/2,-w/2,w/2,w/2]';
    gm = [R1];
    ns = char('R1')';
    sf = 'R1';
    g = decsg(gm,sf,ns);
    geometryFromEdges(model,g);
```

Define Membrane Boundary

Code

$$-\nabla\cdot\left(\mathbf{c}_1\nabla\eta\right)+a_1\eta=f_1$$

$$-\nabla\cdot\left(\mathbf{c}_2\nabla u_s\right)+a_2u_s=f_2$$

```
    k = sprintf('%d*cos(pi*x/%d)',alpha,ell);
    ks = sprintf('-(%d/%d)*pi*sin(pi*x/%d)',alpha,ell,ell);
    mu = sprintf('1./(%d*%d*sqrt(ux(1,:).^2 + (1 - %s.*u(1,:)).^2.*(1 + uy(1,:).^2) ))',↙
ls,ly,k);
    gamma = sprintf('%d*%d*%d*(%d.^2)./(2*%d*%d)',ls,ly,eps,Phi,C1,H);

    c1 = sprintf('%d^2.*(1 - (%d.^2).*(%s.^4)) - %s',ls,ly,mu,gamma);
    c2 = sprintf('%d^2.*(1 - (%d.^2).*(%s.^4).*(1 - %s.*u(1,:)).^2) - %s.*(1 - %s.*u↙
(1,:)).^2',ly,ls,mu,k,gamma,k);
    c3 = sprintf('%d^2',ls);
    c4 = sprintf('%d^2',ly);

    a1 = sprintf('(%d^2)*(%s).^2.*(1 - (%d.^2).*(%s.^4).*(1 + uy(1,:)).^2) - %s.*(%s.^2).↙
*(1 + uy(1,:).^2)',ls,k,ly,mu,gamma,k);
    a2 = sprintf('(%d^2)*(%s).^2',ls,k);

    f1 = sprintf('(%d^2)*(%s.*(1 + 2*ux(2,:) - (%d.^2).*(%s.^4).*(1 + uy(1,:)).^2) + %s.↙
*u(2,:)) - %s.*(%s.^2).*(1 + uy(1,:).^2)',ls,k,ly,mu,ks,gamma,k);
    f2 = sprintf('-(%d^2)*(%s.*u(1,:) + 2*%s.*ux(1,:))',ls,ks,k);

    c = char(c1,'0','0',c2,c3,'0','0',c4);
    a = char(a1,a1);
    f = char(f1,f2);
```

Code

Create Mesh

```
applyBoundaryCondition(model,'Edge',1:model.Geometry.NumEdges,...
    'u',[0,0],'Vectorized','on');
mesh = generateMesh(model);
```

Solve PDE

```
u = pdenonlin(model,c,a,f,'Tol',1e-8,'U0',u0);
u0 = u; % Update initial guess
numNodes = size(model.Mesh.Nodes,2);
```

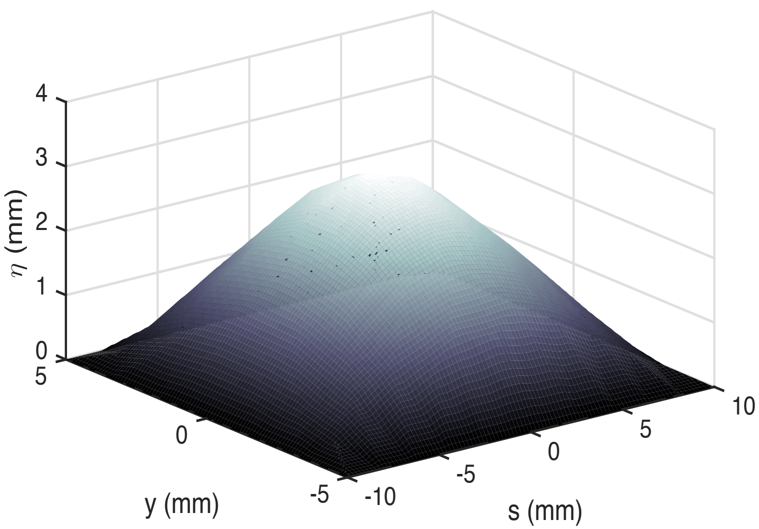
Extract solutions (and their derivatives)

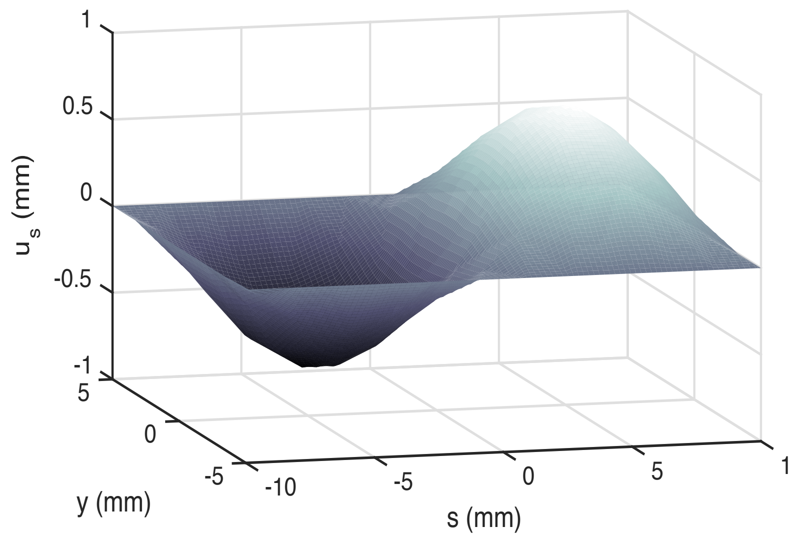
```
n = 100;
s = linspace(-ell/2,ell/2,n);
ds = ell/(n-1);
y = linspace(-w/2,w/2,n);
dy = w/(n-1);
kappa = alpha*cos(pi*s/ell);

[p,e,t] = meshToPet(mesh);
[ux,uy] = pdegrad(p,t,u);

eta = tri2grid(p,t,u,s,y);
eta_s = tri2grid(p,t,ux(1,:)',s,y);
eta_y = tri2grid(p,t,uy(1,:)',s,y);
us = tri2grid(p,t,u((numNodes+1):2*numNodes),s,y);
us_s = tri2grid(p,t,ux(2,:)',s,y);
us_y = tri2grid(p,t,uy(2,:)',s,y);
```

Results

$$\ell = 2 \text{ cm},\ w = 1 \text{ cm},\ H = 130\ \mu\text{m},$$
$$E = 170 \text{ kPa},\ C_1 = E/6,\ \epsilon = \epsilon_r\epsilon_0,\ \epsilon_r = 2,$$
$$E_f = 1 \text{ MPa},\ w_f = 1 \text{ cm},\ h_f = 1.3 \text{ mm},$$

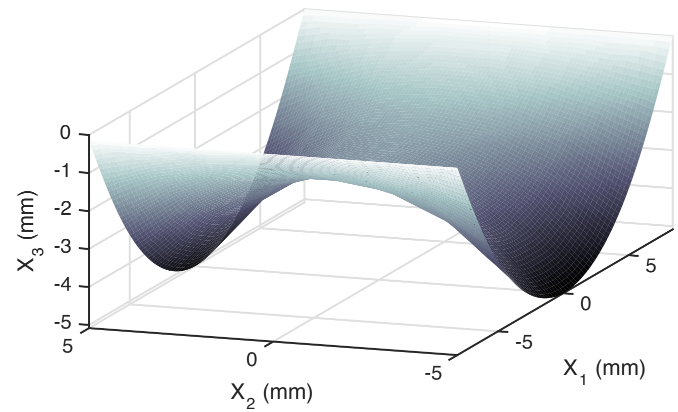
$\Phi = 0$, $\lambda_s = 2$, and $\lambda_y = 1$



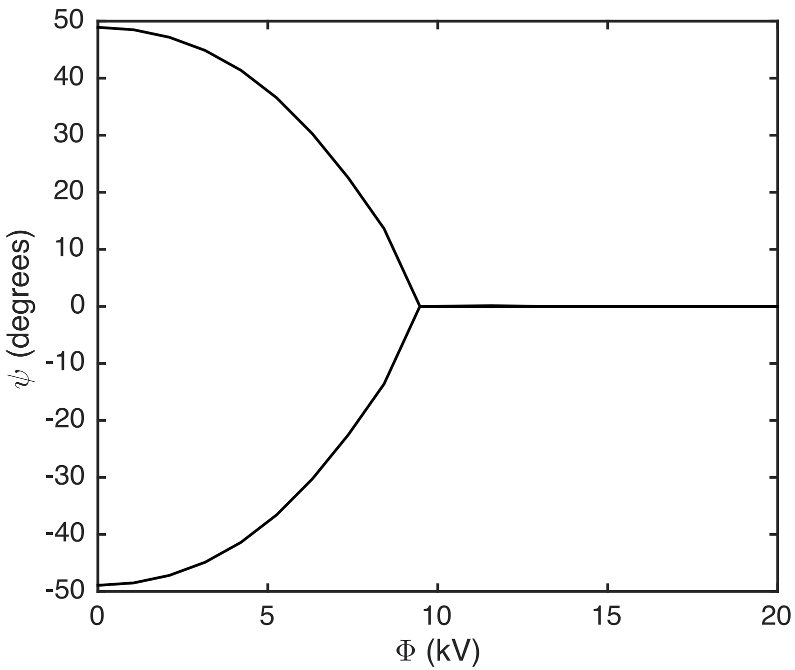


Results

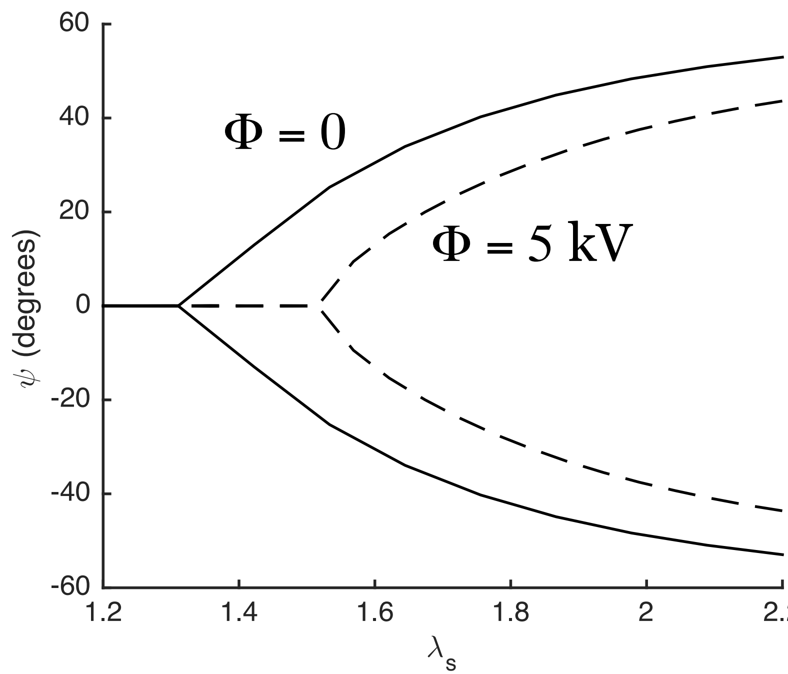
$\ell = 2$ cm, $w = 1$ cm, $H = 130$ μm,
$E = 170$ kPa, $C_1 = E/6$, $\epsilon = \epsilon_r \epsilon_0$, $\epsilon_r = 2$,
$E_f = 1$ MPa, $w_f = 1$ cm, $h_f = 1.3$ mm,



$\lambda_s = 2$, varying Φ



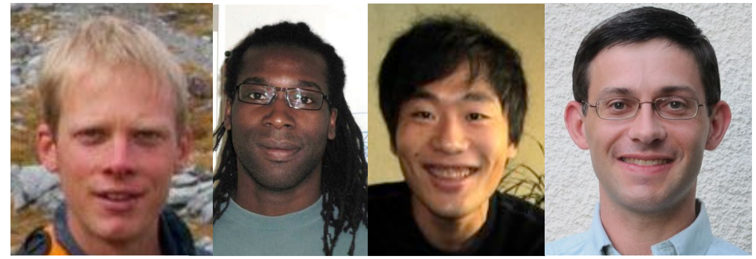
varying λ_s



Simplified DEMES Model

Model and design of dielectric elastomer minimum energy structures

S. Rosset, O. A. Araromi, J. Shintake, H. R. Shea, *Smart Materials & Structures* (2014)



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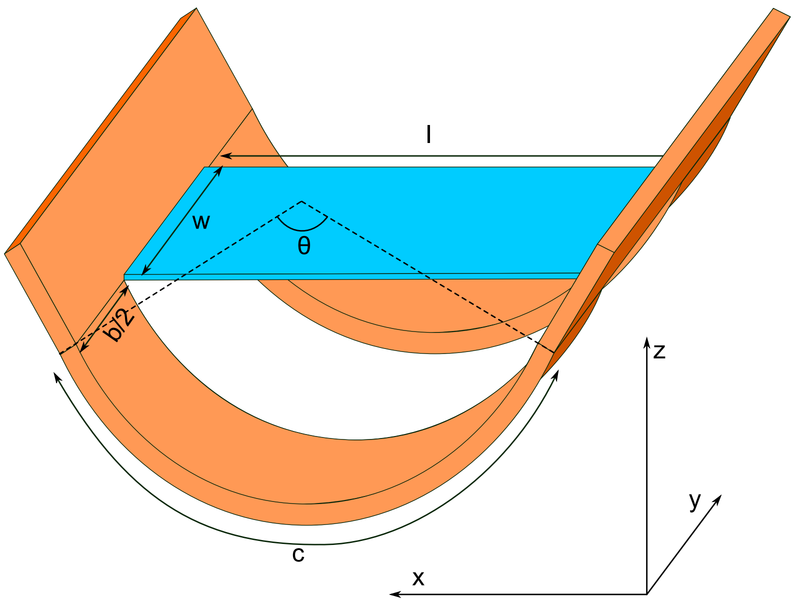


Assumptions:

- Membrane detached along side
- Frame (modulus = Y_f) bends into a pure circle
- Membrane = Gent solid with shear modulus μ.

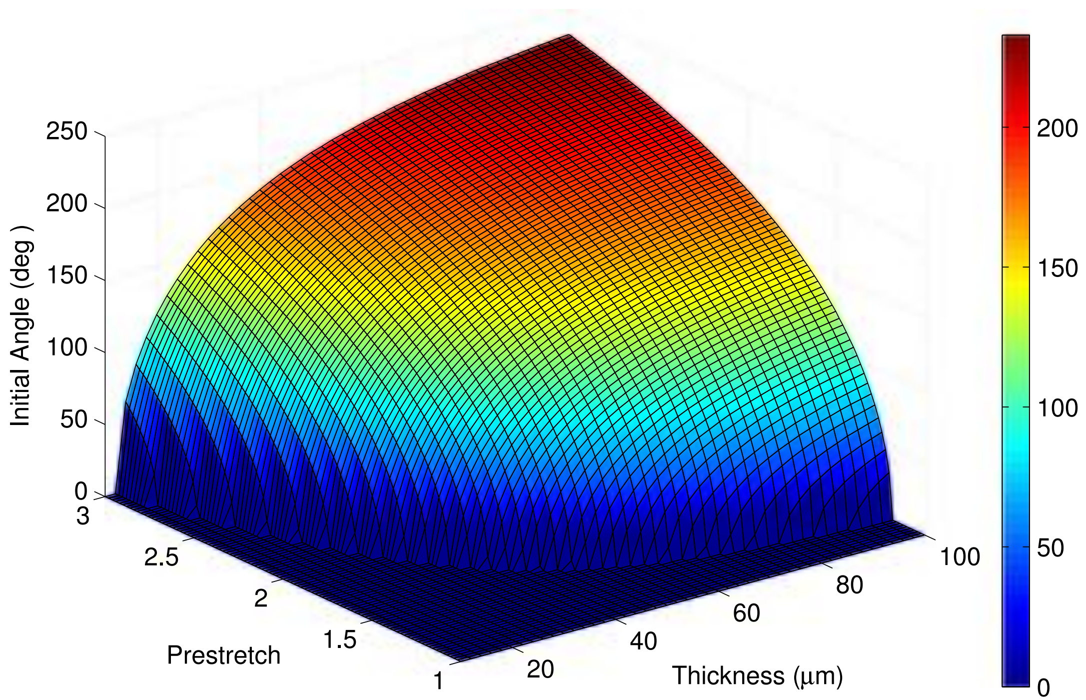
$$U_{tot}(\theta, V) = \frac{1}{2}\frac{Y_f \cdot b \cdot d^3}{12 \cdot c}\theta^2 - \frac{c \cdot w \cdot t_0}{\lambda_p}\frac{\mu J_m}{2}$$

$$\times \ln\left(1 - \frac{\lambda^2 + \lambda^{-2} - 2}{J_m}\right) - \frac{\epsilon \cdot c \cdot w \cdot V^2\lambda^2}{2t_0\lambda_p},$$

Simplified DEMES Model

$$U_{tot}(\theta, V) = \frac{1}{2}\frac{Y_f \cdot b \cdot d^3}{12 \cdot c}\theta^2 - \frac{c \cdot w \cdot t_0}{\lambda_p}\,\frac{\mu J_m}{2} \times \ln\left(1 - \frac{\lambda^2 + \lambda^{-2} - 2}{J_m}\right) - \frac{\epsilon \cdot c \cdot w \cdot V^2 \lambda^2}{2 t_0 \lambda_p}$$

$$\frac{\partial U_{tot}}{\partial \theta} = 0$$

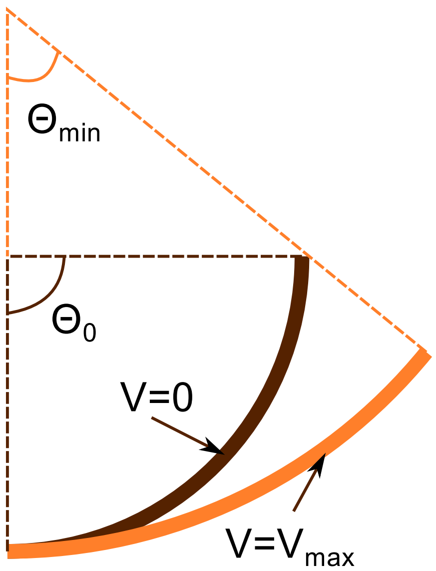


No Voltage

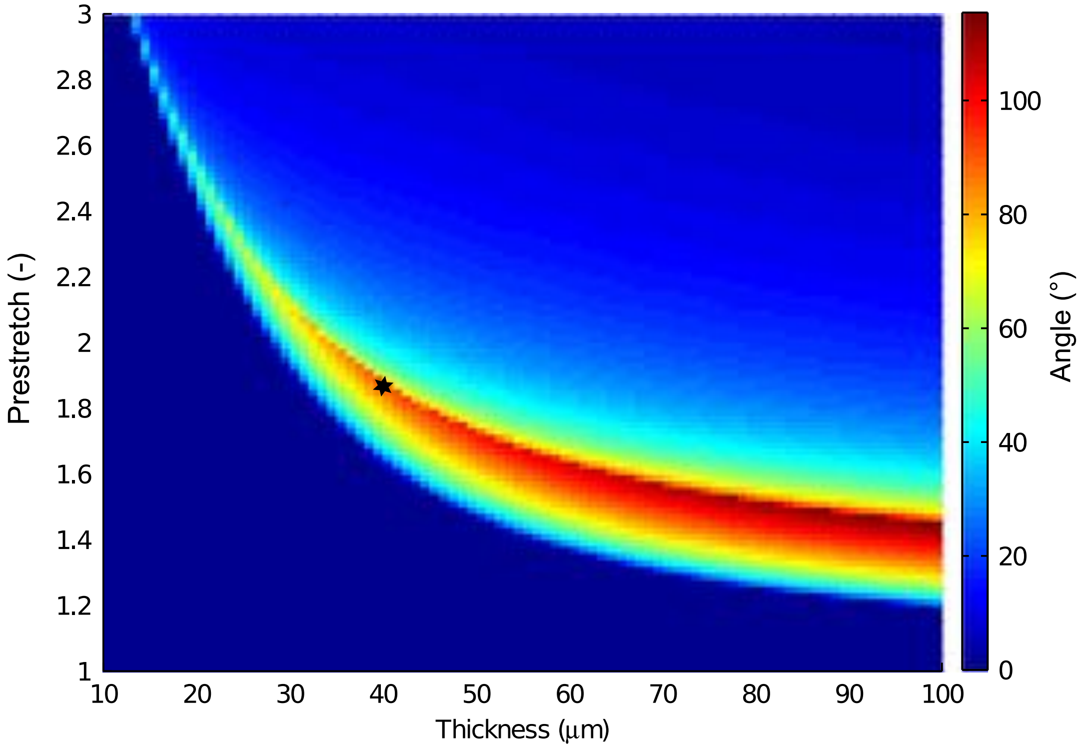
Influence of membrane thickness and prestretch

Simplified DEMES Model

$$U_{tot}(\theta, V) = \frac{1}{2}\frac{Y_f \cdot b \cdot d^3}{12 \cdot c}\theta^2 - \frac{c \cdot w \cdot t_0}{\lambda_p}\frac{\mu J_m}{2} \times \ln\left(1 - \frac{\lambda^2 + \lambda^{-2} - 2}{J_m}\right) - \frac{\epsilon \cdot c \cdot w \cdot V^2 \lambda^2}{2 t_0 \lambda_p}$$



$E_b = 70$ kV/mm



Creep

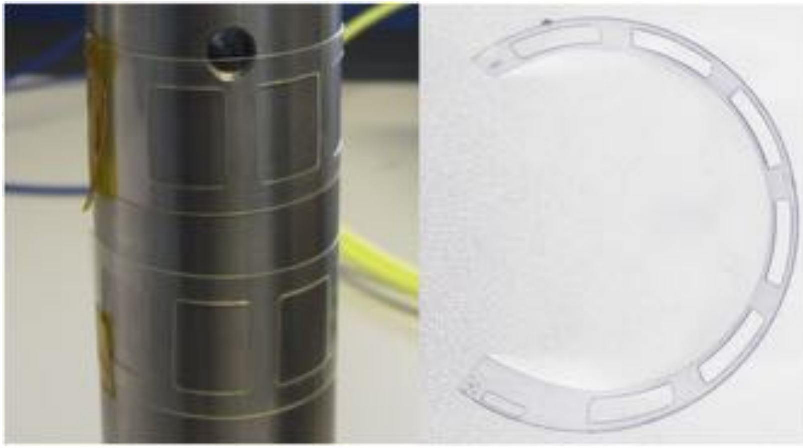


Figure 10. PET frames rolled around a post and left for one night at 80° (left) exhibited a significant creep when released from the post (right).